Yakov G. Sinai, Abel Prize Laureate 2014

CHAOS

Chaos as a phenomenon of daily life is something everybody has experienced. For mathematicians it has been important to understand the deeper meaning of this concept, and how to quantify chaotic behavior.

The term chaos has its origin from the greek term $\chi\acute{\alpha}o\varsigma$, and has been interpreted as "a moving, formless mass from which the cosmos and the gods originated". A more up-to-date definition of the term is something like a state of complete confusion and disorder, with no immediate view of achieving stability.

We have at least two kinds of chaos. A random system will in many cases appear to us as chaotic. Throwing a dice may result in a sequence 3, 1, 5, 3, 3, 2, 6, 1, ... , for which we are sure to find no pattern. Total unpredictability is often considered as chaos.

Another kind of chaos is what is denoted **deterministic chaos**. Deterministic is more or less synonymous with predictable, and deterministic chaos may therefore seem to be somewhat paradoxical. But the chaotic behavior stems from the fact that the system is sensitively dependent on its initial state. As an example, consider the following setup. Onto a rather big sphere we drop small spheres, always trying to hit the top of the bigger sphere. The smaller spheres jump or roll in different directions, depending on which side of the top point they land. The chaotic behavior is a result of small differences in the initial state, i.e. the landing point. This is a deterministic chaotic situation, deterministic because the smaller spheres just obey the physical laws of motion, and chaotic because of the sensitive initial state dependency.

Another example of deterministic chaos is the three-body problem. This problem concerns the trajectories of three bodies, which mutually influence each others' motion, due to gravitational forces. The system is deterministic because every single movement can be predicted using the physical laws of motion, and it is chaotic because of its sensitive dependency on the initial state. This dependency is often denoted the butterfly effect, referring to the theoretical example of a hurricane's formation being contingent on whether or not a distant butterfly has flapped its wings several weeks earlier.

Even the apparently random like system of throwing dice is in fact deterministic. Fixing the initial position and velocity of the dice, the precise shape of the dice and taking into account our accurate knowledge of the surface of the table, we are able to predict the result of a throw of a dice, at least theoretically. But if we impose a small change in an input parameter, we are lost. So even if the system is deterministic, it appears to us as being stochastic.

Let us illustrate some variations of a dynamical system using a marble and a pan. We put the marble in the pan. The initial state of the system is the position of the marble, and the dynamical system gives an accurate description of the trajectories of the marble. The marble will obviously move towards the lowest point of the pan. After some oscillation it will finally reach the equilibrium point. In this dynamical system all trajectories converge to the same point. If we perform the same experiment with a marble on a plane surface, the trajectories will not converge to one specific point, but spread out rectilinearly in all directions. A small change in the initial angle will cause an increasing distance between the trajectories, but the growth of the distance will be constant.

The mathematical notion for measuring this sort of dispersion is Lyapunov's exponent. In the plane example Lyapunov's exponent is 0. In the pan, with all trajectories ending in the same point, Lyapunov's exponent is negative. The most interesting case is when Lyapunov's exponent is positive. In this case trajectories may disperse radically, even if their initial states are very close. Lyapunov's exponent gives a quantification of the rate of dispersion. The French mathematician Jacques Hadamard described in 1898 a dynamical system where Lyapunov's exponent is everywhere positive. Thus the dynamical system shows chaotic behavior everywhere. It is said that Hadamard discovered chaos, at least that he was the first to formally describe a chaotic dynamical system.

The connection between Kolomogorov-Sinai-entropy and Lyapunov's exponent is given in the so-called Pesin's Theorem. A consequence of Pesin's Theorem is that if the entropy is positive, then there exists positive Lyapunov's exponent, and vice versa. The result is by no means obvious. The positive Lyapunov's exponent tells us that trajectories may diverge rapidly, even if their initial states are rather close. Positiv Kolomogorov-Sinai-entropy indicates that the system as a whole shows a certain degree of uncertainty. Pesin's theorem says that the two ways of measuring chaotic behavior of a dynamical system are equivalent.