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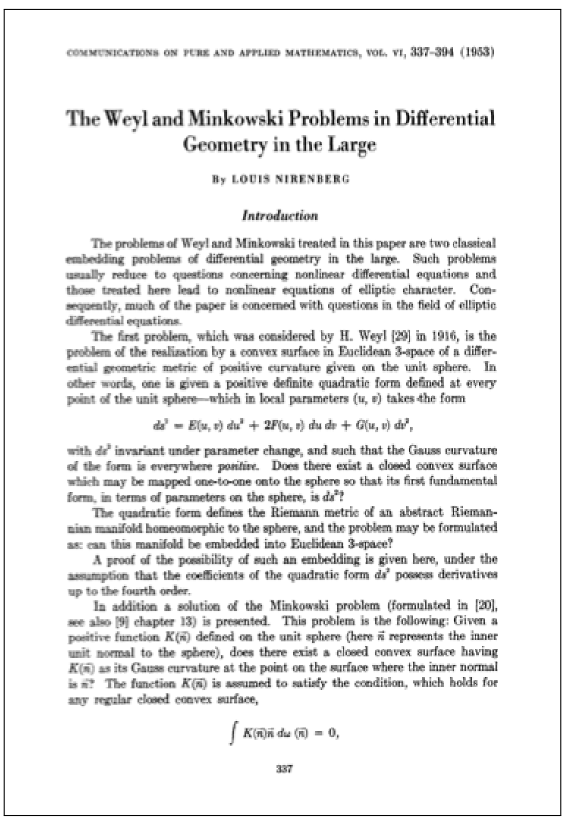
THE MINKOWSKI AND WEYL PROBLEMS

In 1953 Luis Nirenberg published the paper *The Weyl and Minkowski Problem in Differential Geometry in the Large*. In this paper he solved two longstanding open problems in differential geometry, the Minkowski problem raised by Hermann Minkowski in the paper *Volumen und Oberfläche* published in 1903 and the Weyl problem, after Herman Weyl, in the paper *Über die Bestimmung einer geschlossenen konvexen Fläche durch ihr Linienelement* from 1916.



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The Weyl and Minkowski Problems in Differential Geometry in the Large

By LOUIS NIRENBERG

Introduction

The problems of Weyl and Minkowski treated in this paper are two classical embedding problems of differential geometry in the large. Such problems usually reduce to questions concerning nonlinear differential equations and those treated here lead to nonlinear equations of elliptic character. Consequently, much of the paper is concerned with questions in the field of elliptic differential equations.

The first problem, which was considered by H. Weyl [29] in 1916, is the problem of the realization by a convex surface in Euclidean 3-space of a differential geometric metric of positive curvature given on the unit sphere. In other words, one is given a positive definite quadratic form defined at every point of the unit sphere—which in local parameters (u, v) takes the form

$$ds^2 = E(u, v)\, du^2 + 2F(u, v)\, du\, dv + G(u, v)\, dv^2,$$

with ds^2 invariant under parameter change, and such that the Gauss curvature of the form is everywhere *positive*. Does there exist a closed convex surface which may be mapped one-to-one onto the sphere so that its first fundamental form, in terms of parameters on the sphere, is ds^2?

The quadratic form defines the Riemann metric of an abstract Riemannian manifold homeomorphic to the sphere, and the problem may be formulated as: can this manifold be embedded into Euclidean 3-space?

A proof of the possibility of such an embedding is given here, under the assumption that the coefficients of the quadratic form ds^2 possess derivatives up to the fourth order.

In addition a solution of the Minkowski problem (formulated in [20], see also [9] chapter 13) is presented. This problem is the following: Given a positive function $K(\vec{n})$ defined on the unit sphere (here $\vec{n}$ represents the inner unit normal to the sphere), does there exist a closed convex surface having $K(\vec{n})$ as its Gauss curvature at the point on the surface where the inner normal is $\vec{n}$? The function $K(\vec{n})$ is assumed to satisfy the condition, which holds for any regular closed convex surface,

$$\int K(\vec{n})\vec{n}\, d\omega\, (\vec{n}) = 0,$$

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The Minkowski Problem (1903). Given a strictly positive real function f defined on S^2, find a strictly convex compact surface $\Sigma \subset \mathbb{R}^3$ such that the Gauss curvature of Σ at the point x equals $f(\mathbf{n}(x))$, where $\mathbf{n}(x)$ denotes the normal to Σ at x.

Hermann Minkowski (1864-1909)

Consider a sphere of radius R with center C. The sphere is a convex surface, thus any ray starting in the center intersects the sphere in exactly one point. The curvature in the intersection point is $\frac{1}{R^2}$, as it is in every other point on the sphere. Minkowski suggested that we define a function f on the sphere, everywhere positive. An example of such a function is the average annual rainfall on the surface of the earth. This function is illustrated by a colouring of the surface, where e.g. red means dry weather, and dark blue is the colour of the rainy areas.

Now Minkowski asked if we can construct a new surface, still convex, and such that the curvature κ in any direction $\mathbf{n}$ is precisely the value of the given function f in that diresction, i.e. $\kappa(\mathbf{n}) = f(\mathbf{n})$. In the red areas of the sphere, where the value of the "rainfall function" is less than the constant curvature $\frac{1}{R^2}$ we have to "flatten" out the sphere, and in the blue areas it should be more curved. The total curvature of the whole sphere is 4π, so the average value of the function will give the "size" of the new surface.

The Weyl Problem (1916). Consider a two-sphere S^2 and suppose g is a Riemannian metric on S^2 whose Gauss curvature is everywhere positive. Does there exist a global C^2 isometric embedding $X : (S^2, g) \to (\mathbb{R}^3, \sigma)$, where σ is the standard flat metric in $\mathbb{R}^3$?

Hermann Weyl (1885-1955)

The Weyl problem is closely ralated to the Minkowski problem. Weyl defines a Riemannian metric on a sphere. A metric is a more general concept than a function. As a function it is also defined in each point on the sphere, but not by one single value only. A metric assigns a specific value to every direction in any point on the sphere. We illustrate the concept by an example. Consider the surface as a part of a landscape and introduce a metric that expresses the differences in walking speed from various directions in a given point. In a marsh the values are low compared to the values on dry ground. Different values in different directions from a given point reflect variations in the ground in different directions. The distance between two points in this metric is the time spent when walking between the points. A plane with the Euclidean metric has zero Gaussian curvature, while the same plane equipped with the *how-difficult-is-it-to-walk*-metric given above might be rather curved. The Weyl problem asks if given such a metric on a sphere, can we deform the sphere such that, on the deformed sphere, the ordinary distance corresponds to the distance measured by the metric?

Figure 1: The different size of the parts of the body reflects the density of neurons. (Source: Natural History Museum, London)

An illustration of the ideas of the Minkowski and the Weyl problem is the following example: Neurons are not evenly distributed in the human body. Some parts of the body, like our hands, our face and our tongue are much more sensitive to sensations than other parts. The body has the highest density of neurons in those parts. The Euclidean metric, named after the ancient Greek mathematician Euclid, measures ordinary distances between points and area of any region of a surface. If you think of the neuron density metric as the abstract metric and the human body as the 2-sphere, then the weird body in figure 1 illustrates the positive answer to Weyl's question. The different sizes of the various body parts correspond to the neuron density.

Nirenberg, with his fundamental embedding theorems for the sphere S^2 in $\mathbb{R}^3$, having prescribed Gauss curvature or Riemannian metric, solved the classical problems of Minkowski and Weyl (the latter being also treated, simultaneously, by Pogorelov). These solutions were important, both because the problems were representative of a developing area, and because the methods created were the right ones for further applications. Nirenberg's solution of the Minkowski problem was a milestone in global geometry.