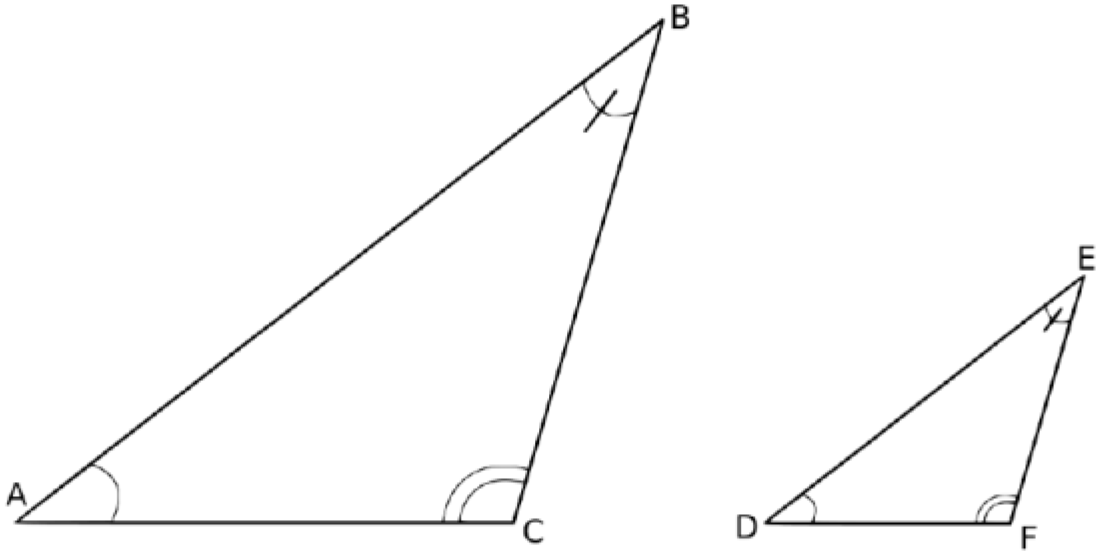
The Abel Prize Winner 2013 Pierre Deligne

Popular version by Arne B. Sletsjøe

Moduli of Stable Curves

With David Mumford, Deligne introduced the notion of an algebraic stack to prove that the moduli space of stable curves is compact. An algebraic stack is an abstract generalisation of the concept of a geometric space.

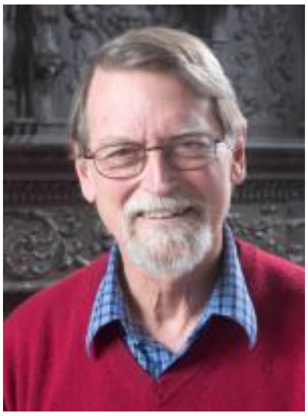
So-called **moduli**-problems has to do with classification of mathematical objects. In general, classification means to collect individuals or objects in families and subfamilies, in a hierarchical way. A moduli problem is about grouping mathematical objects in equivalence classes. The equivalence relation may vary, e.g. we can classify triangles up to similarity, or use the even finer classification, where triangles belong to the same family if they are congruent. But a moduli problem also requires that the classifying objects themselves have rich mathematical structure.



Two similar triangles

To solve the moduli problem for curves we need to find a geometric object, denoted **the universal family**, a smaller geometric object, called the **moduli space**, and a good mapping from the universal family into the moduli space. The moduli space is constructed such that every point in the space corresponds to a certain class of curves, and vice versa, every class

of curves is represented by a point in the moduli space. The universal family contains all the curves and the map from the universal family to the moduli space maps a curve into the point representing the curve. The moduli space is the classifying space of all curves, and the existence of a universal family makes sure that the classification remembersthe structure of the curves. Continuity of the map from the universal family into the moduli space garantees that curves which are almost similar will correspond to points in the moduli space which are close.
Even if we can find a moduli space it is not at all obvious that there exists a universal family. Consider the following example: A rather coarse classification of the real numbers is to split them into to classes, 0 and different from 0. The first class contains one element (0), and the second contains the rest of the real numbers (≠0). The moduli space consists of two distinct points, and a candidate for the universal family is the real numbers itself. But in this case the map from the universal family into the moduli space fails to be continuous, since there are points different from 0, but infinitesimal close to 0. The concusion is that there is a moduli space, but it is not possible to construct a universal family on top of it.



David Mumford

The aim of the work of Deligne and Mumford was to prove that the moduli space of stable curves is compact. Compactness has a technical definition which we explain through an illustrating example. The set of real numbers is non-compact, but a circle is compact. To be able to prove compactness, Deligne and Mumford needed some insight in the universal family.

The problem we face when we try to construct a universal family for stable curves, is the existence of symmetries of the curves, and that different curves may have different number of symmetries. Once again we shall jump to a more intuitve example to illustrate the problems caused by the symmetries. In stead of curves we consider triangles.

Suppose we have constructed a moduli space for triangles, up to similarity. The majority of triangles have no non-trivial symmetries, but isosceles triangles have one and equilateral triangles have five non-trivial symmetries. Using the non-trivial symmetries we can construct a non-constant family, over the circle, of equilateral triangles. During one lap the triangle is turned 120 degrees. We can visualize this family as a soft triangular cylinder which is bent and then glued to become a triangular torus. Just before we glue, we twist one end of the cylinder by 120 degrees. Since the triangles at the two endpoints are supposed to be equilateral, they will still fit. But the twist makes this family non-constant.

The image of the family into the universal family is constant, since the triangle is the same for all points in the circle. A necessary requirement for a universal family is that it is universal, i.e. that it containsevery other family. In this example this is not the case, thus violating the existence of a universal family.

The solution Deligne and Mumford gave to the general moduli question, was to introduce the notion of an algebraic stack, later called a DM stack (or Deligne-Mumford stack). In contrast to the ordinary moduli space, a DM stack includes information about the symmetries of the triangles. If we try to build a universal family over the moduli stack, rather than over the moduli space, we are more likely to succeed since the solution to our problem is forced to be part of the structure of the objects we are working with.