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The Work of Niels Henrik Abel

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During his short life, N.-H. Abel devoted himself to several topics characteristic of the mathematics of his time. We note that, after an unsuccessful investigation of the influence of the Moon on the motion of a pendulum, he chose subjects in pure mathematics rather than in mathematical physics. It is possible to classify these subjects in the following way:

1. solution of algebraic equations by radicals;
2. new transcendental functions, in particular elliptic integrals, elliptic functions, abelian integrals;
3. functional equations;
4. integral transforms;
5. theory of series treated in a rigourous way.

The first two topics are the most important and the best known, but we shall see that there are close links between all the subjects in Abel's treatment. As the first published papers are related to subjects 3 and 4, we will begin our study with functional equations and the integral transforms.

1 Functional Equations

In the year 1823, Abel published two norwegian papers in the first issue of *Magasinet for Naturvidenskaberne*, a journal edited in Christiania by Ch. Hansteen. In the first one, titled *Almindelig Methode til at finde Funktioner af een variabel Størrelse, naar en Egenskab af disse Funktioner er udtrykt ved en Ligning mellom to Variable* (*Œuvres*, t. I, p. 1–10), Abel considers a very general type of functional equation: $V(x, y, \varphi\alpha, f\beta, F\gamma, \dots, \varphi'\alpha, f'\beta, F'\gamma, \dots) = 0$, where $\varphi, f, F, \dots$ are unknown functions in one variable and $\alpha, \beta, \gamma, \dots$ are known functions of the two independent variables x, y. His method consists in successive eliminations of the unknown $\varphi, f, F, \dots$ between the given equation $V = 0$ and the equations obtained by differentiating this equation with α constant, then with β constant, etc. If, for instance $\alpha = \text{const}$, there is a relation between x and y, and y may be considered as a function of x and the constant value of α; if n is the highest order of derivative of φ present in V, it is possible to eliminate $\varphi\alpha$ and its derivatives by differentiating V $n+1$ times with α constant. We then eliminate $f\beta$ and its derivative, and so on, until we arrive at a differential equation with only one unknown function of one variable. Naturally, all the functions, known and unknown, are tacitly supposed indefinitely differentiable.

Abel applies this to the particular case $\varphi\alpha = f(x, y, \varphi\beta, \varphi\gamma)$, where f, α, β and γ are given functions and φ is unkown; he gets a first order differential equation with respect to φ. For instance, the functional equation of the logarithm $\log xy = \log x + \log y$ corresponds to the case where $\alpha(x, y) = xy$, $\beta(x, y) = x$, $\gamma(x, y) = y$ and $f(x, y, t, u) = t + u$; differentiating with $xy = \text{const}$, we get $0 = x\varphi'x - y\varphi'y$, from which, with $y = \text{const}$, we get $\varphi'x = \frac{c}{x}$, where $c = y\varphi'y$. In the same way, the functional equation for arctangent,

$$\arctan\frac{x+y}{1-xy} = \arctan x + \arctan y,$$

corresponds to $\alpha(x, y) = \frac{x+y}{1-xy}$, $\beta(x, y) = x$, $\gamma(x, y) = y$ and $f(x, y, t, u) = t + u$; differentiating with α constant gives $0 = (1 + x^2)\varphi'x - (1 + y^2)\varphi'y$, whence $\varphi'x = \frac{c}{1+x^2}$ if $c = (1 + y^2)\varphi'y$.

When $\beta(x, y) = x$, $\gamma(x, y) = y$ and $f(x, y, t, u) = t \cdot u$, we get first

$$\varphi y \cdot \varphi' x \frac{\partial\alpha}{\partial y} - \varphi x \cdot \varphi' y \frac{\partial\alpha}{\partial x} = 0,$$

whence $\frac{\varphi'x}{\varphi x}$ as a known function of x if y is supposed constant. For $\alpha(x, y) = x + y$, this gives $\frac{\varphi'x}{\varphi x} = c = \frac{\varphi'y}{\varphi y}$, so $\log\varphi x = cx$(for $\varphi(0) = 1$) and $\varphi x = e^{cx}$; for $\alpha(x, y) = xy$, $\frac{\varphi'x}{\varphi x} = \frac{c}{x}$, so $\log\varphi x = c\log x$ ($\varphi(1) = 1$) and $\varphi x = x^c$.

All these examples were classical as is the next one, coming from mechanics. The law of composition of two equal forces making an angle $2x$ leads to the functional equation

$$\varphi x \cdot \varphi y = \varphi(x+y) + \varphi(x-y); \tag{1}$$

where φx is the ratio of the resultant force to one of the two equal forces. Differentiating with $y + x = \text{const}$, one gets $\varphi' x \cdot \varphi y - \varphi x \cdot \varphi' y = 2\varphi'(x-y)$; another differentiation, with $x - y = \text{const}$, gives $\varphi'' x \cdot \varphi y - \varphi x \cdot \varphi'' y = 0$. If y is regarded as constant, this gives $\varphi'' x + c\varphi x = 0$ and $\varphi x = \alpha \cos(\beta x + \gamma)$ with α, β and γ constant. From (1), one sees that $\alpha = 2$ and $\gamma = 0$ and the problem imposes $\varphi\left(\frac{\pi}{2}\right) = 0$, so $\beta = 1$ and $\varphi(x) = 2\cos x$.

Here is another case of application of Abel's general method: the equation has the form $\psi\alpha = F(x, y, \varphi x, \varphi' x, \ldots, fy, f' y, \ldots)$, where α is a given function of x and y and φ, f, ψ are unknown functions. By differentiating with α constant, one gets a relation between $x, \varphi x, \varphi' x, \ldots$ and $y, fy, f' y, \ldots$, whence two differential equations, with respect to φ and to f, considering successively y and x as constant; if φ and f are determined, it is easy to determine ψ by the functional equation. In particular, if $\psi(x+y) = \varphi x \cdot f' y + fy \cdot \varphi' x$, so that $\alpha(x, y) = x + y$, the differentiation with α constant gives $\varphi x \cdot f'' y - fy \cdot \varphi'' x = 0$, and $\varphi x = a \sin(bx + c)$, $fy = a' \sin(by + c)$ then $\psi(x+y) = aa'b \sin(b(x+y) + c + c')$ so that $\psi\alpha = aa'b \sin(b\alpha + c + c')$.

In the case of $\psi(x+y) = f(xy) + \varphi(x-y)$, one gets $0 = f'(xy)(y-x) + 2\varphi'(x-y)$. Abel takes $xy = c$ as constant and writes $\varphi'\alpha = k\alpha$, where $\alpha = x - y$ and $k = \frac{f'(c)}{2}$, so $\varphi\alpha = k' + \frac{k}{2}\alpha^2$; then he takes $x - y = c$ constant and writes $f'\beta = c' = \frac{2\varphi' c}{c}$, so $f\beta = c'' + c'\beta$. Finally

$$\psi(x+y) = c'' + c'xy + k' + \frac{k}{2}(x-y)^2$$

or $\psi\alpha = c'' + c'x(\alpha - x) + k' + \frac{k}{2}(2x-\alpha)^2 = c'' + \frac{k}{2}\alpha^2 + k' + x\alpha(c' - 2k) + (2k - c')x^2$ and we see that the condition $c' = 2k$ is necessary; $\psi\alpha = k' + c'' + \frac{k}{2}\alpha^2$.

The third example is $\varphi(x+y) = \varphi x \cdot fy + fx \cdot \varphi y$, which gives

$$0 = \varphi' x \cdot fy - \varphi x \cdot f' y + f' x \cdot \varphi y - fx \cdot \varphi' y; \tag{2}$$

if one supposes that $f(0) = 1$ and $\varphi(0) = 0$, one gets $0 = \varphi' x - \varphi x \cdot c + fx \cdot c'$ by making $y = 0$ ($c = f'(0)$ and $c' = -\varphi'(0)$); so $fx = k\varphi x + k'\varphi' x$ and, substituting this value in (2) and making y constant: $\varphi'' x + a\varphi' x + b\varphi x = 0$ etc.

Abel returned to the study of functional equations in the paper "*Recherche des fonctions de deux quantités variables indépendantes x et y, telles que* $f(x, y)$*, qui ont la propriété que* $f(z, f(x, y))$ *est une fonction symétrique de* z, x *et* y", published in German in the first volume of Crelle's *Journal* in 1826 (*Œuvres*, t. I, p. 61–65). The condition of the title characterises a composition law which is associative and commutative; it may be written as $f(x, y) = f(y, x)$, $f(z, f(x, y)) = f(x, f(y, z)) = f(y, f(z, x))$ or

$$f(z, r) = f(x, v) = f(y, s) \tag{3}$$

if $f(x, y) = r$, $f(y, z) = v$ and $f(z, x) = s$. Differentiating with respect to x, to y and to z and multiplying the results, one gets

$$\frac{\partial r}{\partial x}\frac{\partial v}{\partial y}\frac{\partial s}{\partial z} = \frac{\partial r}{\partial y}\frac{\partial v}{\partial z}\frac{\partial s}{\partial x}. \tag{4}$$

But, by the definition of v, the quotient of $\frac{\partial v}{\partial y}$ by $\frac{\partial v}{\partial z}$ is a function φy when z is regarded as constant; in the same manner, φx is the quotient of $\frac{\partial s}{\partial x}$ by $\frac{\partial s}{\partial z}$, so (4) becomes $\frac{\partial r}{\partial x}\varphi y = \frac{\partial r}{\partial y}\varphi x$ and this gives r as an arbitrary function ψ of $\Phi(x) + \Phi(y)$, where Φ is a primitive of φ. So $f(x, y) = \psi(\Phi(x) + \Phi(y))$; putting this expression in (3) and making $\Phi z = \Phi y = 0$ and $\Phi x = p$, one gets $\Phi\psi p = p + c$, where $c = \Phi\psi(0)$, and then $\Phi f(x, y) = \Phi(x) + \Phi(y) + c$ or

$$\Psi f(x, y) = \Psi(x) + \Psi(y) \tag{5}$$

where $\Psi(x) = \Phi(x)+c$. In other words, Abel finds that f is conjugate to the addition law by the function Ψ: he has determined the one-parameter groups.

The second volume of Crelle's *Journal* (1827) (*Œuvres*, t. I, p. 389–398) contains another paper of Abel on a functional equation:

$$\varphi x + \varphi y = \psi(xfy + yfx) = \psi(r), \tag{6}$$

where $r = xfy + yfx$; this equation includes, as particular cases, the laws of addition for log ($fy = \frac{1}{2}y$, $\varphi x = \psi x = \log x$) and for arcsin ($fy = \sqrt{1 - y^2}$, $\varphi x = \psi x = \arcsin x$). One has $\varphi' x = \psi' r \cdot \frac{\partial r}{\partial x}$, $\varphi' y = \psi' r \cdot \frac{\partial r}{\partial y}$, so $\varphi' x \cdot \frac{\partial r}{\partial y} = \varphi' y \cdot \frac{\partial r}{\partial x}$ or

$$\varphi' y \cdot (fy + yf'x) = \varphi' x \cdot (fx + xf'y), \tag{7}$$

whence, for $y = 0$,

$$a\alpha - \varphi' x \cdot (fx + \alpha' x) = 0, \tag{8}$$

where $a = \varphi'(0)$, $\alpha = f(0)$ and $\alpha' = f'(0)$, a differential equation which determines φ if f is known. Substituting in (7), one gets $(fx + \alpha' x)(fy + yf'x) = (fy + \alpha' y)(fx + xf'y)$ or

$$\frac{1}{y}(\alpha' fy - fy \cdot f'y - \alpha' yf'y) = \frac{1}{x}(\alpha' fx - fx \cdot f'x - \alpha' xf'x) = m,$$

necessarily constant. So

$$f'x \cdot (fx + \alpha' x) + (mx - \alpha' fx) = 0, \tag{9}$$

which determines f; as this differential equation is homogenenous, it is easily integrated by putting $fx = xz$, in the form $\log c - \log x = \frac{1}{2}\log(z^2 - n^2) + \frac{\alpha'}{2n}\log\frac{z-n}{z+n}$, where $m = -n^2$ and c is a constant of integration. One gets

$$c^{2n} = (fx - nx)^{n+\alpha'}(fx + nx)^{n-\alpha'},$$

with $c = \alpha$, then φ by (8) and (6) is verified if $\psi x = \varphi\left(\frac{x}{\alpha}\right) + \varphi(0)$. Abel explicitly treats the case in which $n = \alpha' = \frac{1}{2}$: $fx = \alpha + \frac{1}{2}x$, then $\varphi x = a\alpha \log(\alpha + x) + k$ and $\psi x = 2k + a\alpha \log(\alpha^2 + x)$.

The relation $\alpha^{2n} = (fx - nx)^{n+\alpha'}(fx + nx)^{n-\alpha'}$, which determines f, allows us to express $fx - nx$, and then x and fx, in terms of $fx + nx = v$; turning back to (8), this gives $\varphi x = \frac{a\alpha}{n+\alpha'} \log(cnx + c\,fx)$. When $n = 0$, the relation which determines f takes the form $e^{\alpha' x} = \left(\frac{fx}{\alpha}\right)^{fx}$ and we have $\varphi x = \frac{a\alpha}{\alpha'} \log c\alpha + \frac{a\alpha x}{fx}$, $\psi x = \frac{2a\alpha}{\alpha'} \log c\alpha + \frac{ax}{f\left(\frac{x}{\alpha}\right)}$. The equation (6) signifies that $\alpha f\left(\frac{x\,fy + y\,fx}{\alpha}\right) = fx \cdot fy$ and Abel verifies that it is satisfied. Another particular case is that in which $\alpha' = \infty$. When m is finite, (9) reduces to $x f'x - fx = 0$, so that $fx = cx$; when m is infinite and equals $-p\alpha'$, (9) becomes $x fx - px - fx = 0$ and $fx = px \log cx$. In this last case, one gets by (7) $y\varphi' y - x\varphi' x = 0$, whence $x\varphi' x = k$ constant and $\varphi x = k \log mx$ (a new m) and then $\psi(pv \log c^2 v) = k \log m^2 v$.

A memoir left unpublished by Abel is devoted to the equation $\varphi x + 1 = \varphi(fx)$, where f is given and φ unknown (*Œuvres*, t. II, p. 36–39, mem. VI). Abel introduces a function ψ such that $f\psi y = \psi(y + 1)$; one may take ψ arbitrarily on the interval $[0, 1]$ and define ψ on $[0, +\infty[$ by $\psi(y + n) = f^n(\psi y)$ (and on $]-\infty, 0]$ by $\psi(y - n) = f^{-n}(\psi y)$ if f is bijective). For $x = \psi y$, the functional equation becomes $1 + \varphi\psi y = \varphi\psi(y + 1)$, so that $\varphi\psi y = y + \chi y$ where χ is any periodic function of y with period 1. Denoting the inverse function of ψ by $\grave{\psi}$, Abel gets

$$\varphi x = \grave{\psi} x + \chi(\grave{\psi} x).$$

As an example, he takes $fx = x^n$ and $\psi y = a^{n^y}$, so that $\grave{\psi} x = \frac{\log\log x - \log\log a}{\log n}$ and

$$\varphi x = \frac{\log\log x - \log\log a}{\log n} + \chi\left(\frac{\log\log x - \log\log a}{\log n}\right),$$

for instance $\varphi x = \frac{\log\log x}{\log n}$ if $\chi = 0$ and $a = e$.

Abel treats in a similar manner the general equation $F(x, \varphi(fx), \varphi(\psi x)) = 0$, where F, f and ψ are given functions and φ is unknown. Supposing that $fx = y_t$ and $\psi x = y_{t+1}$ or $y_{t+1} = \psi(`fy_t)$, one has $F(`fy_t, u_t, u_{t+1}) = 0$, where $u_t = \varphi y_t$; this difference equation has a solution $u_t = \theta t$ and $\varphi z = \theta(`y_t)$. For instance the equation $(\varphi x)^2 = \varphi(2x) + 2$ leads to $(u_t)^2 = u_{t+1} + 2$ and, if $u_1 = a + \frac{1}{a}$, this gives $u_t = a^{2^{t-1}} + \frac{1}{a^{2^{t-1}}}$; on the other hand $y_{t+1} = 2y_t$, so that $y_t = c \cdot 2^{t-1}$ (c constant) and $2^{t-1} = \frac{x}{c}$. Finally, $\varphi x = b^x + b^{-x}$ ($b = a^{1/c}$). As we see, this type of equations is treated with a method different from the preceeding one, by reduction to finite difference equation.

Another type of functional equation is related to the *dilogarithm*

$$\psi x = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \ldots + \frac{x^n}{n^2} + \ldots,$$

which Abel studies in the posthumous memoir XIV (*Œuvres*, t. II, p. 189–193) after Legendre's *Exercices de Calcul intégral*. The study is based on the summation of the series (for $|x| \leq 1$) in the form of an integral

$$\psi x = -\int_0 \frac{dx}{x} \log(1-x) \tag{10}$$

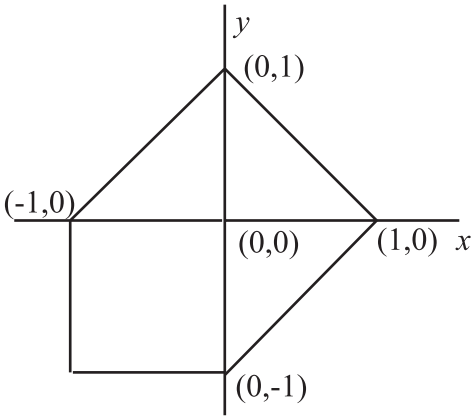
and Abel reproduces several functional equations given by Legendre, as for example

$$\psi x + \psi(1-x) = \frac{\pi^2}{6} - \log x \cdot \log(1-x).$$

But he adds a remarkable new property:

$$\psi\left(\frac{x}{1-x}\cdot\frac{y}{1-y}\right) = \psi\left(\frac{y}{1-x}\right) + \psi\left(\frac{x}{1-y}\right) - \psi y - \psi x - \log(1-y)\log(1-x) \tag{11}$$

for (x, y) in the interior domain of the figure



In order to prove (11), Abel substitutes $\frac{a}{1-a}\cdot\frac{y}{1-y}$ for x in (10):

$$\begin{aligned}
\psi\left(\frac{a}{1-a}\cdot\frac{y}{1-y}\right) &= -\int\left(\frac{dy}{y}+\frac{dy}{1-y}\right)\log\frac{1-a-y}{(1-a)(1-y)} \\
&= -\int\frac{dy}{y}\log\left(1-\frac{y}{1-a}\right) + \int\frac{dy}{y}\log(1-y) \\
&\quad -\int\frac{dy}{1-y}\log\left(1-\frac{a}{1-y}\right) + \int\frac{dy}{1-y}\log(1-a) \\
&= \psi\left(\frac{y}{1-a}\right) - \psi y - \int\frac{dy}{1-y}\log\left(1-\frac{a}{1-y}\right) \\
&\quad -\log(1-a)\log(1-y),
\end{aligned}$$

where the remaining integral is computed by taking $z = \frac{a}{1-y}$ as variable:

$$\int \frac{dy}{1-y} \log\left(1 - \frac{a}{1-y}\right) = \int \frac{dz}{z} \log(1-z) = -\psi z = -\psi\left(\frac{a}{1-y}\right) + \text{const.}$$

The constant of integration is determined by taking $y = 0$ and is found to be ψa.

Abel was the first mathematician to give a general and (almost) rigourous proof of Newton's famous binomial formula

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \frac{m(m-1)(m-2)}{2 \cdot 3}x^3 + \dots \tag{12}$$

He published his demonstration in the first volume of *Crelle's Journal* (1826, *Recherches sur la série* $1 + mx + \frac{m(m-1)}{2}x^2 + \frac{m(m-1)(m-2)}{2 \cdot 3}x^3 + \dots$, *Œuvres*, t. I, p. 218–250). He uses an idea of Euler, already exploited by Lagrange and Cauchy: writing $\varphi(m)$ the second member of (12), one proves that

$$\varphi(m+n) = \varphi(m)\varphi(n), \tag{13}$$

so that $\varphi(m) = A^m = (1+x)^m$ for m rational as was observed by Euler. Lagrange extended this proof to every value of m admitting that φ is an analytic function of m. Cauchy used an analogous strategy for m real and $|x| < 1$ using the continuity of φ, for which his proof was unfortunately incomplete. Abel considers the most general case, with x and m complex, with $|x| < 1$ or $|x| = 1$ and $\operatorname{Re} m > -1$ (if $x = -1$, one needs $\operatorname{Re} m > 0$).

For $m = k + k'i$, $\varphi(m) = f(k, k')(\cos\psi(k, k') + i \sin\psi(k, k'))$, with f, ψ continuous functions of k, k' real. The continuity is almost established by Abel in his theorem V, but this theorem is not entirely correct. The concept of uniform convergence did not exist at that time and it was not easy to give a general theorem for the continuity of the sum of a series of continuous functions. The functional equation (13) becomes

$$\begin{aligned} f(k+\ell, k'+\ell') &= f(k, k') f(\ell, \ell'); \\ \psi(k+\ell, k'+\ell') &= 2m\pi + \psi(k, k') + \psi(\ell, \ell'), \end{aligned} \tag{14}$$

where m is an integer, which must be constant because of the continuity of ψ. In a first step, Abel treats the functional equation for ψ; putting $\theta k = \psi(k, k'+\ell') = 2m\pi + \psi(k, k') + \psi(0, \ell')$ he gets

$$\theta k + \theta\ell = a + \theta(k+\ell), \tag{15}$$

with $a = 2m\pi + \psi(0, k') + \psi(0, \ell')$, whence

$$\theta k = ck + a, \tag{16}$$

where c is a function of k', ℓ'. Indeed, taking $\ell = k, 2k, \dots, \rho k$ in (15) and adding the results, Abel gets $\rho\theta k = (\rho-1)a + \theta(\rho k)$ and $\theta\rho = \rho(\theta(1) - a) + a$ for $k = 1$,

ρ a natural integer; then, for $k = \frac{\mu}{\rho}$ ($\mu, \rho \in \mathbf{N}, \rho \neq 0$), $\rho\theta\left(\frac{\mu}{\rho}\right) = (\rho - 1)a + \theta\mu$ and $\theta\left(\frac{\mu}{\rho}\right) = c\frac{\mu}{\rho} + a$, with $c = \theta(1) - a$. This formula is extended to the negative values of k using $\theta(-k) = 2a - \theta k$ and, by continuity, to every real value of k. So

$$\psi(k, k' + \ell') = ck + 2m\pi + \psi(0, k') + \psi(0, \ell'), \tag{17}$$

where $c = \theta(k', \ell')$, a function of k' and ℓ'. For $k = 0$, this gives

$$\psi(0, k' + \ell') = 2m\pi + \psi(0, k') + \psi(0, \ell'),$$

a functional equation which may be treated as (15) and which has the solution

$$\psi(0, k') = \beta' k' - 2m\pi,$$

with an arbitrary constant β'; then (17) becomes

$$\psi(k, k' + \ell') = \theta(k', \ell') \cdot k + \beta'(k' + \ell') - 2m\pi,$$

also equal to $2m\pi + \psi(k, k') + \psi(0, \ell') = \psi(k, k') + \beta'\ell'$ by (14), so that $\psi(k, k') = Fk' \cdot k + \beta' k' - 2m\pi$, with $Fk' = \theta(k', \ell')$ independent of ℓ' and $F(k' + \ell') = Fk' = F(0) = \beta$ a constant. Finally

$$\psi(k, k') = \beta k + \beta' k' - 2m\pi. \tag{18}$$

To treat the functional equation (14) for f, Abel writes $f(k, k') = e^{F(k,k')}$ and $F(k + \ell, k' + \ell') = F(k, k') + F(\ell, \ell')$, a functional equation analog to that for ψ with $m = 0$, so its solution is of the form $F(k, k') = \delta k + \delta' k'$, with two arbitrary constant δ, δ'. Finally

$$\varphi(k + k'i) = e^{\delta k + \delta' k'}(\cos(\beta k + \beta' k') + i \sin(\beta k + \beta' k')) \tag{19}$$

and it remains to determine the constants β, β', δ and δ'.

For $k = 1$ and $k' = 0$, $\varphi(1) = 1 + x = 1 + \alpha\cos\phi + i\alpha\sin\phi$, where $\alpha = |x| < 1$ and $\phi = \arg x$; this gives $e^{\delta}\cos\beta = 1 + \alpha\cos\phi$ and $e^{\delta}\sin\beta = \alpha\sin\phi$, so that

$$e^{\delta} = (1 + 2\alpha\cos\phi + \alpha^2)^{\frac{1}{2}} \quad \text{and} \quad \tan\beta = \frac{\alpha\sin\phi}{1 + \alpha\cos\phi}, \beta = s + \mu\pi, \tag{20}$$

with $-\frac{\pi}{2} \leq s \leq \frac{\pi}{2}$ and $\mu \in \mathbf{Z}$. Now, for $k' = 0$ and any k, let $p = f\alpha$ and $q = \theta\alpha$ designate the real and the imaginary part of the series $\varphi(k)$, which are continuous functions of α after Abel's theorem IV (which is correct); one has

$$f\alpha = e^{\delta k}\cos ks\cos k\mu\pi - e^{\delta k}\sin ks\sin k\mu\pi,$$
$$\theta\alpha = e^{\delta k}\sin ks\cos k\mu\pi + \varepsilon^{\delta\kappa}\cos ks\sin k\mu\pi$$

and $\cos k\mu\pi = e^{-\delta k}(f\alpha\cdot\cos ks + \theta\alpha\cdot\sin ks)$, $\sin k\mu\pi = e^{-\delta k}(\theta\alpha\cdot\cos ks - f\alpha\cdot\sin ks)$, independent of α by continuity. For $\alpha = 0$, $e^{\delta} = 1$ and $s = 0$ after (19) whereas $f\alpha = 1$ and $\theta\alpha = 0$, so $k\mu\pi = 0$ and

$$f\alpha = (1 + 2\alpha\cos\phi + \alpha^2)^{\frac{k}{2}}\cos ks, \quad \theta\alpha = (1 + 2\alpha\cos\phi + \alpha^2)^{\frac{k}{2}}\sin ks; \qquad (21)$$

this is Cauchy's result for $f\alpha + i\theta\alpha = |1 + x|^k(\cos ks + i\sin ks) = (1 + x)^k$.

Abel now considers the case in which $m = in$ is purely imaginary; then the series (12) is convergent for any value of n by d'Alembert's rule (which is Abel's theorem II) and Abel states its continuity as a function of n as a consequence of his theorem V. He writes the real and imaginary parts of the series in the form

$$p = 1 + \lambda_1\alpha\cos\theta_1 + \ldots + \lambda_\mu\alpha^\mu\cos\theta_\mu + \ldots$$
$$\text{and} \quad q = \lambda_1\alpha\sin\theta_1 + \ldots + \lambda_\mu\alpha^\mu\sin\theta_\mu + \ldots,$$

where $\lambda_\mu = \delta_1\delta_2\ldots\delta_\mu$, $\theta_\mu = \mu\phi + \gamma_1 + \gamma_2 + \ldots + \gamma_\mu$ and

$$\frac{ni - \mu + 1}{\mu} = \delta_\mu(\cos\gamma_\mu + i\sin\gamma_\mu).$$

From (19) he knows that $p = e^{\delta' n}\cos\beta' n$ and $q = e^{\delta' n}\sin\beta' n$; in order to determine δ' resp. β', he takes the limits of $\frac{e^{\delta' n}\cos\beta' n - 1}{n}$ resp. $\frac{e^{\delta' n}\sin\beta' n}{n}$ for $n = 0$. As $\delta_\mu \to \frac{\mu-1}{\mu}$ and $\gamma_\mu \to \pi$ ($\mu \geq 2$; for $\mu = 1$, $\gamma_1 = \frac{\pi}{2}$), he gets $\frac{\lambda_\mu}{n} \to \frac{1}{\mu}$ and $\gamma_\mu \to \mu(\phi + \pi) - \frac{\pi}{2}$ so

$$\beta' = \alpha\cos\phi - \frac{1}{2}\alpha^2\cos 2\phi + \frac{1}{3}a\alpha^3\cos 3\phi - \ldots,$$
$$\delta' = -\alpha\sin\phi + \frac{1}{2}\alpha^2\sin 2\phi - \frac{1}{3}\alpha^3\sin 3\phi + \ldots$$

Now, computing in the same manner the limits, for $k = 0$, of $\frac{f\alpha - 1}{k}$ and $\frac{\theta\alpha}{k}$, one gets from (21)

$$\delta = \alpha\cos\phi - \frac{1}{2}\alpha^2\cos 2\phi + \frac{1}{3}\alpha^3\cos 3\phi - \ldots \qquad (22)$$
$$\text{and} \quad \beta = \alpha\sin\phi - \frac{1}{2}\alpha^2\sin 2\phi + \frac{1}{3}\alpha^3\sin 3\phi - \ldots,$$

so that $\beta' = \delta$ and $\delta' = -\beta$. The sum (19) of the series (12) for $m = k + k'i$ is

$$e^{\delta k - \beta k'}(\cos(\beta k + \delta k') + i\sin(\beta k + \delta k'))$$

with β and δ as in (20). Let us interpret Abel's result: writing $\delta + i\beta = \log(1 + x)$, one gets

$$m\log(1 + x) = (k + ik')(\delta + i\beta) = k\delta - k'\beta + i(k\beta + k'\delta),$$

so that $\varphi(m) = (1 + x)^m$.

Comparing (20) and (22), Abel gets

$$\frac{1}{2}\log(1 + 2\alpha\cos\phi + \alpha^2) = \alpha\cos\phi - \frac{1}{2}\alpha^2\cos 2\phi + \frac{1}{3}\alpha^3\cos 3\phi - \ldots$$

and

$$\arctan \frac{\alpha \sin \phi}{1+\alpha \cos \phi} = \alpha \sin \phi - \frac{1}{2}\alpha^2 \sin 2\phi + \frac{1}{3}\alpha^3 \sin 3\phi - \dots ; \qquad (23)$$

by making α tend toward ± 1, $\frac{1}{2}\log(2\pm 2\cos\phi) = \pm\cos\phi - \frac{1}{2}\cos 2\phi \pm \frac{1}{3}\cos 3\phi - \dots$ and $\frac{1}{2}\phi = \sin\phi - \frac{1}{2}\sin 2\phi + \frac{1}{3}\sin 3\phi - \dots$ for $-\pi < \phi < \pi$. If $\phi = \frac{\pi}{2}$ and $-1 \le \alpha \le 1$ in (23), one gets Gregory's series $\arctan\alpha = \alpha - \frac{1}{3}\alpha^3 + \frac{1}{5}\alpha^5 - \dots$

Taking $x = i\tan\phi$ and m real in the binomial series, Abel's finds

$$\cos m\phi = (\cos\phi)^m \left(1 - \frac{m(m-1)}{1\cdot 2}(\tan\phi)^2 + \frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}(\tan\phi)^4 - \dots\right),$$
$$\sin m\phi = (\cos\phi)^m \left(m\tan\phi - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(\tan\phi)^3 + \dots\right)$$

for $-\frac{\pi}{4} \le \phi \le \frac{\pi}{4}$ (for $\phi = \pm\frac{\pi}{4}$, m must be > -1).

Now, taking $|x| = 1$ and $m > -1$, he finds as the real part of

$$(1+x)^m(\cos\alpha - i\sin\alpha):$$

$$\cos\alpha + \frac{m}{1}\cos(\alpha-\phi) + \frac{m(m-1)}{1\cdot 2}\cos(\alpha - 2\phi) + \dots$$
$$= (2+2\cos\phi)^{\frac{m}{2}}\cos\left(\alpha - \frac{m\phi}{2} + m\rho\pi\right)$$

where ρ is an integer such that $|\phi - 2\rho\pi| \le \pi$ (with the restriction $m > 0$ in case of equality). The substitutions $\phi = 2x$ and $\alpha = mx, mx + \frac{\pi}{2}, m\left(x+\frac{\pi}{2}\right)$ or $m\left(x+\frac{\pi}{2}\right) - \frac{\pi}{2}$ give Abel various formulae, for instance

$$(2\cos x)^m \cos 2m\rho\pi = \cos mx + \frac{m}{1}\cos(m-2)x + \frac{m(m-1)}{1\cdot 2}\cos(m-4)x + \dots$$
$$(2\cos x)^m \sin 2m\rho\pi = \sin mx + \frac{m}{1}\sin(m-2)x + \frac{m(m-1)}{1\cdot 2}\sin(m-4)x + \dots$$

for $2\rho\pi - \frac{\pi}{2} \le x \le 2\rho\pi + \frac{\pi}{2}$. Abel was the first to prove rigourously such formulae for m non integer; in a letter to his friend Holmboe (16 January 1826, *Œuvres*, t. II, p. 256), he states his result and alludes to the unsuccessful attempts of Poisson, Poinsot, Plana and Crelle.

Other examples of functional equations in Abel's work may be mentioned, as the famous Abel theorem (see §5), which may be interpreted in this way. In a letter to Crelle (9 August 1826, *Œuvres*, t. II, p. 267), Abel states his theorem for genus 2 in a very explicit manner: he considers the hyperelliptic integral $\varphi(x) = \int \frac{(\alpha+\beta x)dx}{\sqrt{P(x)}}$ where P is a polynomial of degree 6; then Abel's theorem is the functional equation $\varphi(x_1) + \varphi(x_2) + \varphi(x_3) = C - (\varphi(y_1) + \varphi(y_2))$, where x_1, x_2 and x_3 are independant variables, C is a constant and y_1, y_2 are the roots of the equation

$$y^2 - \left(\frac{c_2^2 + 2c_1 - a_4}{2c_2 - a_5} - x_1 - x_2 - x_3\right) y + \frac{\frac{c^2-a}{x_1x_2x_3}}{2c_2 - a_5} = 0,$$

with $P(x) = a + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + x^6$ and $c + c_1x_j + c_2x_j^2 + x_j^3 = \sqrt{P(x_j)}$ for $j = 1, 2, 3$. Abel says that this functional equation completely characterises the function φ.

Abel discovered how to express the elliptic functions as quotients of two entire functions of the type of Weierstrass' σ-function; there is an allusion to that in the introduction to his *Précis d'une théorie des fonctions elliptiques*, published in the fourth volume of *Crelle's Journal* (1829, *Œuvres*, t. I, p. 527–528) and in a letter to Legendre (25 November 1828, *Œuvres*, t. II, p. 274–275). The elliptic function $\lambda(\theta)$ is defined by

$$\theta = \int_0^{\lambda\theta} \frac{dx}{\Delta(x, c)}, \quad \text{where} \quad \Delta(x, c) = \pm\sqrt{(1 - x^2)(1 - c^2x^2)},$$

and $\lambda\theta = \frac{\varphi\theta}{f\theta}$ where the entire functions φ and f are solutions of the system of functional equations $\varphi(\theta' + \theta) \cdot \varphi(\theta' - \theta) = (\varphi\theta \cdot f\theta')^2 - (\varphi\theta' \cdot f\theta)^2$, $f(\theta' + \theta) \cdot f(\theta' - \theta) = (f\theta \cdot f\theta')^2 - c^2(\varphi\theta \cdot \varphi\theta')^2$. This system is partially solved in a notebook of 1828, with x and y in place of θ' and θ; supposing φ odd and f even and taking the second derivative with respect to x at $x = 0$, Abel finds the equations $f''y + fy - (f'y)^2 = a(fy)^2 - c^2b(\varphi y)^2$ and $-\varphi''y + \varphi y + (\varphi' y)^2 = b(fy)^2 - a(\varphi y)^2$ with $a = f(0) \cdot f''(0)$ and $b = (\varphi'0)^2$. If it is supposed that $a = 0$ and $b = 1$, this reduces to $(f'y)^2 - f''y \cdot fy = c^2(\varphi y)^2$, $(\varphi' y)^2 - \varphi''y \cdot \varphi y = (fy)^2$. Again differentiating four times at $x = 0$, Abel obtains the derivatives of f up to the 4^{th} order and φ, but his computation, aimed to find differential equations for f and φ, stops here.

Two posthumous papers by Abel are devoted to differential equations of Riccati type. In the first one, *Sur l'équation différentielle $dy + (p + qy + ry^2)dx = 0$, où p, q et r sont des fonctions de x seul* (*Œuvres*, t. II, p. 19–25), Abel shows how to transform this equation in another one of the form $dy + (P + Qy^2)dx = 0$. Two methods are proposed. The first one, by putting $y = z + r'$ with $r' = -\frac{q}{2r}$, which gives $dz + (P + Qz^2)dx = 0$ with $P = p - \frac{q^2}{4r} - \frac{dq}{dx}\frac{1}{2r} + \frac{dr}{dx}\frac{q}{2r^2}$ and $Q = r$.

The second one, which is classical, by putting $y = zr'$ with $r' = e^{-\int qdx}$; this gives $P = pe^{\int qdx}$ and $Q = re^{-\int qdx}$. Abel observes that when $pe^{\int qdx} = are^{-\int qdx}$ or $e^{\int qdx} = \sqrt{\frac{ar}{p}}$, the equation, which is written $dy + \left(p + \frac{1}{2}\left(\frac{dr}{rdx} - \frac{dp}{pdx}\right) y + ry^2\right) dx = 0$, may be integrated in finite terms, giving $y = -\sqrt{\frac{p}{r}} \tan\left(\int \sqrt{rp}dx\right)$. For example, the equation $dy + \left(\frac{1}{x} - \frac{y^2}{x}\right) dx = 0$ has a solution of the form $y = \frac{1-cx^2}{1+cx^2}$ and the equation $dy + \left(x^m + \frac{1}{2}(n - m)\frac{y}{x} + x^ny^2\right) dx = 0$ has a solution of the form $y = -x^{\frac{m-n}{2}} \tan\left(c + \frac{2}{m+n+2}x^{\frac{1}{2}(m+n+2)}\right)$; in the case in which $n = -m - 2$, this solution becomes $y = -x^{m+1}\tan(\log k'x)$. Another easy case of integration is given by the relations $\frac{p}{c} = \frac{q}{2a} = r$; in this case $y = -a + \sqrt{a^2 - c}\,\frac{1+e^{-\frac{2}{c}\sqrt{a^2-c}\int p dx}}{1-e^{-\frac{2}{c}\sqrt{a^2-c}\int p dx}}$.

Abel explains how to solve the equation when a particular solution y' is known. Putting $y = z+y'$, he finds $dz+((q+2ry')z+rz^2)dx = 0$ and $y = y'+\frac{e^{-\int(q+2ry')dx}}{\int e^{-\int(q+2ry')}rdx}$. For example the equation $dy + \left(\frac{1}{x^2} + \frac{ay}{x} + cy^2\right)dx = 0$ has the particular solution $y' = \left(\frac{1-a}{2c} \pm \sqrt{\left(\frac{1-a}{2c}\right)^2 - \frac{1}{c}}\right)\frac{1}{x}$ and this leads to the general solution

$$y = \left(\frac{1-a}{2c} \pm \sqrt{\left(\frac{1-a}{2c}\right)^2 - \frac{1}{c}}\right)\frac{1}{x} + \frac{kx^{-\left(1\pm\sqrt{(1-a)^2-4c}\right)}}{C \pm \frac{ck}{\sqrt{(1-a)^2-4c}}x^{\mp\sqrt{(1-a)^2-4c}}}.$$

Other cases of integration are found by Euler's method of integrating factor: the expression $zdy + z(p+qy^2)dx$ is a complete differential when $\frac{\partial z}{\partial x} = \frac{\partial(z(p+qy^2))}{\partial y}$ or, if $z = e^r$, when $\frac{\partial r}{\partial x} = (p+qy^2)\frac{\partial r}{\partial y} + 2qy$. Abel tries with $r = a\log(\alpha+\beta y)$ with a constant and α, β functions of x only. He finds the conditions $a\alpha' - a\beta p = a\beta' - 2\alpha q = a\beta q + 2\beta q = 0$, where α', β' are the derivatives of α, β. Thus the equation $dy + \left(\frac{\alpha'}{\beta} - \frac{\beta'}{\alpha}y^2\right)dx = 0$ admits the integrating factor $z = \frac{1}{(\alpha+\beta y)^2}$ and the solution $y = -\frac{\alpha}{\beta} + \frac{1}{\beta^2\left(C - \int\frac{\beta'}{\alpha\beta^2}dx\right)}$.

In the second paper, Abel considers the differential equation

$$(y+s)dy + (p+qy+ry^2)dx = 0,$$

which is reduced to the form $zdz + (P+Qz)dx = 0$ by the substitution $y = \alpha + \beta z$ with $\alpha = -s$ and $\beta = e^{-\int rdx}$. One has $P = (p - qs + rs^2)e^{2\int rdx}$ and $Q = \left(q - 2rs - \frac{ds}{dx}\right)e^{\int rdx}$. If $P = 0$, this equation has the solution $z = \int\left(2rs + \frac{ds}{dx} - q\right)e^{\int rdx}dx$ so that the equation

$$(y+s)dy + (qs - rs^2 + qy + ry^2)dx = 0$$

has for solution $y = -s + e^{-\int rdx}\int\left(2rs + \frac{ds}{dx} - q\right)e^{\int rdx}dx$. When $Q = 0$, the equation in z has the solution $z = \sqrt{2\int(qs - p - rs^2)e^{2\int rdx}dx}$ and the equation

$$(y+s)dy + \left(p + \left(2rs + \frac{ds}{dx}\right)y + ry^2\right)dx = 0$$

has for solution $y = -s + e^{-\int rdx}\sqrt{2\int\left(rs^2 - p + \frac{sds}{dx}\right)e^{2\int rdx}dx}$.

In order that $z = e^r$ be an integrating factor for the equation

$$ydy + (p+qy)dx = 0,$$

we must impose $y\frac{\partial r}{\partial x}-(p+qy)\frac{\partial r}{\partial y}-q=0$. For $r=\alpha+\beta y$, this gives the conditions $\frac{d\beta}{dx}=\frac{d\alpha}{dx}-q\beta=p\beta+q=0$, so $\beta=-c$, $\alpha=-c\int qdx$ and $-cp+q=0$. For $r=\alpha+\beta y+\gamma y^2$, one finds $\gamma=c$, $\beta=2c\int qdx$, $q+2cp\int qdx=0$ and $\alpha=2c\int qdx\int qdx-\int\frac{qdx}{\int qdx}$. When $q=1$, we find that the equation $ydy+\left(\frac{1}{c(x+a)}+y\right)dx=0$ admits the integrating factor $\frac{1}{x+a}e^{-\frac{c}{2}(x+y+a)^2}$. More generally, for $r=\alpha+\alpha_1 y+\alpha_2 y^2+\ldots+\alpha_n y^n$, one finds $n+2$ conditions $\frac{d\alpha_n}{dx}=0=\frac{d\alpha_{n-1}}{dx}-nq\alpha_n=\frac{d\alpha_{n-2}}{dx}-(n-1)q\alpha_{n-1}-np\alpha_n=\ldots=\frac{d\alpha}{dx}-q\alpha_1-2p\alpha_2=q+p\alpha_1=0$ for the $n+1$ coefficients α_k; so there is a relation between p and q. For $n=3$, Abel finds

$$q+6cp\int qdx\int qdx+3cp\int pdx=0.$$

A function $r=\frac{1}{\alpha+\beta y}$ leads to the conditions $\frac{d\beta}{dx}+\beta^2 q=\frac{d\alpha}{dx}-\beta q+2\alpha\beta q=\alpha^2 q-\beta p=0$ and the equation $ydy+\left(\left(\frac{C}{(\int qdx)^2}+\frac{1}{2}\right)^2 q\int qdx+qy\right)dx$ admits the integrating factor $e^{\frac{1}{\alpha+\beta y}}$ with $\beta=\frac{1}{\int qdx}$ and $\alpha=\frac{C}{(\int qdx)^2}+\frac{1}{2}$.

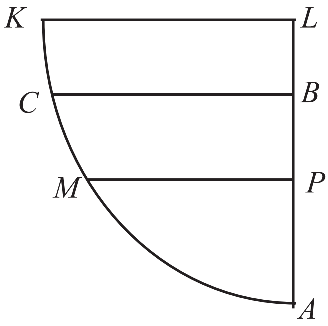
Another form tried by Abel is $r=a\log(\alpha+\beta y)$; he finds that $ydy-\left(\frac{a+1}{a^2}q-qy\right)dx=0$ has the integrating factor $\left(\frac{(a+1)c}{a}\int qdx+cy\right)^a$. More generally $r=a\log(y+\alpha)+a'\log(y+\alpha')$ gives a new form of differential equation integrable by the factor e^r.

2 Integral Transforms and Definite Integrals

The second Norwegian paper of Abel, titled *Opløsning af et Par Opgaver ved Hjelp af bestemte Integraler* (1823, *Œuvres*, t. I, p. 11–27), studies in its first part the integral equation $\psi a=\int\limits_{x=0}^{x=a}\frac{ds}{(a-x)^n}$ where ψ is a given function, s an unknown function of x and $n<1$.

In the case where $n=\frac{1}{2}$, s is interpreted as the length of a curve to be found, along which the fall of a massive point from the height a takes a time equal to ψa. Let the curve be KCA, the initial position of the falling body be the point C and its initial velocity be 0; when the falling body is in M its velocity is proportional to $\sqrt{a-x}$, where a is the total height AB and x is the height AP. So the fall along an infinitesimal arc MM' takes a time dt proportional to $-\frac{ds}{\sqrt{a-x}}$, where $s=AM$ is the curvilineal abscissa along the curve, and the total duration of the fall is proportional to the integral $\int\limits_{x=0}^{x=a}\frac{ds}{\sqrt{a-x}}$.

Abel's equation is probably the first case of an integral equation in the history of mathematics; before that, Euler had introduced in his *Institutiones Calculi Integralis* the general idea to solve a differential equation by a definite integral, for instance by



the so called Laplace transform and Fourier (1811) and Cauchy (1817) had studied the Fourier transform and its law of inversion.

Abel supposes that s has a development in power series with respect to x: $s = \sum \alpha^{(m)} x^m$; differentiating and integrating term by term, he obtains $\psi a = \sum m\alpha^{(m)} \int_0^a \frac{x^{m-1}dx}{(a-x)^n}$. One has $m\int_0^a \frac{x^{m-1}dx}{(a-x)^n} = ma^{m-n}\int_0^1 \frac{t^{m-1}dt}{(1-t)^n} = \frac{\Gamma(1-n)\Gamma(m+1)}{\Gamma(m-n+1)} a^{m-n}$, using the Eulerian function Γ, for which Abel refers to Legendre's *Exercices de Calcul intégral*; so

$$\psi a = \Gamma(1-n) \sum \alpha^{(m)} a^{m-n} \frac{\Gamma(m+1)}{\Gamma(m-n+1)}.$$

Let now $\psi a = \sum \beta^{(k)} a^k$ (ψ is implicitly supposed to be analytic); by identification, Abel gets $\alpha^{(n+k)} = \frac{\Gamma(k+1)}{\Gamma(1-n)\Gamma(n+k+1)} \beta^{(k)} = \frac{\beta^{(k)}}{\Gamma n \cdot \Gamma(1-n)} \int_0^1 \frac{t^k dt}{(1-t)^{1-n}}$, so that

$$\begin{aligned} s = \sum \alpha^{(m)} x^m &= \frac{x^n}{\Gamma n \cdot \Gamma(1-n)} \int_0^1 \frac{\sum \beta^{(k)} (xt)^k dt}{(1-t)^{1-n}} \\ &= \frac{x^n}{\Gamma n \cdot \Gamma(1-n)} \int_0^1 \frac{\psi(xt)dt}{(1-t)^{1-n}} = \frac{x^n \sin n\pi}{\pi} \int_0^1 \frac{\psi(xt)dt}{(1-t)^{1-n}} \end{aligned}$$

and, in the particular case where $n = \frac{1}{2}$, $s = \frac{\sqrt{x}}{\pi} \int_0^1 \frac{\psi(xt)dt}{\sqrt{1-t}}$.

Abel applies this result in the case where $\psi a = ca^n$ (c constant, and the exponent n not to be confused with that of $a-x$ in the general problem, which is now $\frac{1}{2}$), in which $s = Cx^{n+\frac{1}{2}}$, with $C = \frac{c}{\pi} \int_0^1 \frac{t^n dt}{\sqrt{1-t}}$; then $dy = \sqrt{ds^2 - dx^2} = dx\sqrt{kx^{2n-1} - 1}$, where $k = \left(n + \frac{1}{2}\right)^2 C^2$, so

$$y = \int dx \sqrt{kx^{2n-1} - 1} = k' + x\sqrt{k-1}$$

in the particular case where $n = \frac{1}{2}$; in this case, the curve KCA solution of the problem is a straight line. The isochronic case, where $\psi a = c$ constant is another interesting case; here $n = 0$ and $s = C\sqrt{x}$ ($C = \frac{2c}{\pi}$), equation characterising the cycloid. This problem was initially solved by Huygens (1673).

Turning back to the general case Abel gives another interpretation of the solution as a derivative of ψ of non-integral order $-n$. Indeed, if $\psi x = \sum \alpha^{(m)} x^m$ and if k is a natural integer,

$$\frac{d^k \psi}{dx^k} = \sum \alpha^{(m)} \frac{\Gamma(m+1)}{\Gamma(m-k+1)} x^{m-k};$$

in which the right hand side is still meaningfull when k is not a natural integer, and then

$$\frac{\Gamma(m+1)}{\Gamma(m-k+1)} = \frac{1}{\Gamma(-k)} \int_0^1 \frac{t^m dt}{(1-t)^{1+k}},$$

so that the right hand side becomes $\frac{1}{x^k \Gamma(-k)} \int_0^1 \frac{\sum \alpha^{(m)} (xt)^m dt}{(1-t)^k} = \frac{1}{x^k \Gamma(-k)} \int_0^1 \frac{\psi(xt) dt}{(1-t)^k}$, whence the definition of $\frac{d^{-n}\psi}{dx^{-n}} = \frac{x^n}{\Gamma n} \int_0^1 \frac{\psi(xt) dt}{(1-t)^{1-n}}$ and the solution $s = \frac{1}{\Gamma(1-n)} \frac{d^{-n}\psi}{dx^{-n}}$ of the initial problem. The derivative of order n of $s = \varphi x$ is naturally $\frac{1}{\Gamma(1-n)} \psi x$, which means that

$$\frac{d^n \varphi}{da^n} = \frac{1}{\Gamma(1-n)} \int_0^a \frac{\varphi' x dx}{(a-x)^n} \quad (n < 1);$$

for $n = \frac{1}{2}$, $\psi x = \sqrt{\pi} \frac{d^{\frac{1}{2}} s}{dx^{\frac{1}{2}}}$.

The idea of a derivative of non-integral order comes from Leibniz; it was based on the analogy, discovered by Leibniz, between the powers and the differentials in the celebrated formula for $d^n(xy)$, which has the same coefficient as $(x+y)^n = p^n(x+y)$ in Leibniz' notation. The general binomial formula, with exponent e non necessarily integral, suggests to Leibniz a formula for $d^e(xy)$ as an infinite series (letter to the Marquis de l'Hospital, 30 September 1695). Abel's procedure is an extension of a formula given by Euler in 1730: $\frac{d^n(z^e)}{dz^n} = z^{e-n} \frac{\int_0^1 dx(-lx)^e}{\int_0^1 dx(-lx)^{e-n}}$, where e and n are arbitrary numbers and l notes the logarithm. At Abel's time, some other authors also considered derivatives of arbitrary order, as Fourier and Cauchy, but the theory really began with Liouville in 1832 and Riemann in 1847.

At the end of this part, Abel reports that he has solved the more general integral equation $\psi a = \int \varphi(xa) fx \cdot dx$, where ψ and f are given functions and φ is unknown.

Abel published a German version of this study in *Crelle's Journal* (vol. I, 1826, *Œuvres*, t. I, p. 97–101). He finds the solution without any use of power series, starting from the Eulerian integral of the first kind $\int_0^1 \frac{y^{\alpha-1}dy}{(1-y)^n} = \frac{\Gamma\alpha\cdot\Gamma(1-n)}{\Gamma(\alpha+1-n)}$, which gives $\int_0^a \frac{z^{\alpha-1}dz}{(a-z)^n} = \frac{\Gamma\alpha\cdot\Gamma(1-n)}{\Gamma(\alpha+1-n)}a^{\alpha-n}$ and

$$\int_0^x \frac{da}{(x-a)^{1-n}}\int_0^a \frac{z^{\alpha-1}dz}{(a-z)^n} = \frac{\Gamma\alpha\cdot\Gamma(1-n)}{\Gamma(\alpha+1-n)}\int_0^x \frac{a^{\alpha-n}da}{(x-a)^{1-n}}$$
$$= \Gamma n\cdot\Gamma(1-n)\frac{\Gamma\alpha}{\Gamma(\alpha+1)}x^\alpha = \frac{x^\alpha}{\alpha}\Gamma n\cdot\Gamma(1-n).$$

Then, if $fx = \int \varphi\alpha\cdot x^\alpha d\alpha$, one has $\int_0^x \frac{da}{(x-a)^{1-n}}\int_0^a \frac{f'z.dz}{(a-z)^n} = \Gamma n\cdot\Gamma(1-n)\,fx$ and

$$fx = \frac{\sin n\pi}{\pi}\int_0^x \frac{da}{(x-a)^{1-n}}\int_0^a \frac{f'zdz}{(a-z)^n}.$$

Therefore, in the original problem $\varphi a = \int_{x=0}^{x=a} \frac{ds}{(a-x)^n}$, one has

$$\frac{\sin n\pi}{\pi}\int_0^x \frac{\varphi a da}{(x-a)^{1-n}} = \frac{\sin n\pi}{\pi}\int_0^x \frac{da}{(x-a)^{1-n}}\int_0^a \frac{ds}{(a-x)^n} = s.$$

In this paper, there is no mention of derivatives of non-integral order.

The second part of the Norwegian paper is devoted to the proof of the integral formula:

$$\varphi(x+y\sqrt{-1}) + \varphi(x-y\sqrt{-1}) = \frac{2y}{\pi}\int_{-\infty}^{+\infty} e^{-v^2y^2}vdv\int_{-\infty}^{+\infty}\varphi(x+t)e^{-v^2t^2}dt,$$

giving as a particular case $\cos y = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-t^2+\frac{1}{4}\frac{y^2}{t^2}}dt$ when $\varphi t = e^t$, $x = 0$. Abel uses the developments

$$\varphi(x+y\sqrt{-1})+\varphi(x-y\sqrt{-1}) = 2\left(\varphi x - \frac{\varphi'' x}{1\cdot 2}y^2 + \frac{\varphi'''' x}{1\cdot 2\cdot 3\cdot 4}y^4 - \dots\right),$$
$$\varphi(x+t) = \varphi x + t\varphi' x + \frac{t^2}{1\cdot 2}\varphi'' x + \frac{t^3}{1\cdot 2\cdot 3}\varphi''' x + \dots$$

and the definite integrals

$$\int_{-\infty}^{+\infty} e^{-v^2t^2} t^{2n} dt = \frac{\Gamma\left(\frac{2n+1}{2}\right)}{v^{2n+1}}, \quad \int_{-\infty}^{+\infty} e^{-v^2y^2} v^{-2n} dv = \Gamma\left(\frac{1-2n}{2}\right) y^{2n-1}.$$

The last two parts of the paper give summation formulae by means of definite integrals. From the development $\frac{1}{e^t-1} = e^{-t} + e^{-2t} + e^{-3t} + \dots$ and the value $\int_0^\infty e^{-kt} t^{2n-1} dt = \frac{\Gamma(2n)}{k^{2n}}$, Abel deduces $\int_0^\infty \frac{t^{2n-1}dt}{e^t-1} = \Gamma(2n)\zeta(2n)$; the Eulerian formula $\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{\Gamma(2n+1)} A_n$, where A_n is the n-th Bernoulli number, then gives $A_n = \frac{2n}{2^{2n-1}\pi^{2n}} \int_0^\infty \frac{t^{2n-1}dt}{e^t-1} = \frac{2n}{2^{2n-1}} \int_0^\infty \frac{t^{2n-1}dt}{e^{\pi t}-1}$. Using these values in the Euler–MacLaurin sum formula $\sum \varphi x = \int \varphi x.dx - \frac{1}{2}\varphi x + A_1 \frac{\varphi' x}{1 \cdot 2} - A_2 \frac{\varphi''' x}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$ and Taylor series for $\varphi\left(x \pm \frac{t}{2}\sqrt{-1}\right)$, Abel finds

$$\sum \varphi x = \int \varphi x dx - \frac{1}{2}\varphi x + \int_0^\infty \frac{\varphi\left(x + \frac{t}{2}\sqrt{-1}\right) - \varphi\left(x - \frac{t}{2}\sqrt{-1}\right)}{2\sqrt{-1}} \frac{dt}{e^{\pi t} - 1}. \quad (24)$$

This formula was already published in 1820 by Plana in the Memoirs of the Turin Academy; Plana found it by the same type of formal manipulations as Abel. It was rigorously established by Schaar in 1848, using Cauchy's calculus of residues.

As particular applications of this formula, Abel gives the values of some definite integrals: for $\varphi x = e^{mx}$, $\int_0^\infty \frac{\sin mt dt}{e^{\pi t}-1} = \frac{1}{e^m-1} - m + \frac{1}{2}$, for $\varphi x = \frac{1}{x}$,

$$\int_0^\infty \frac{tdt}{\left(x^2 + \frac{1}{4}t^2\right)(e^{\pi t} - 1)} = 2\log x - \frac{1}{x} - 2\sum \frac{1}{x} + 3 + \int_0^\infty \frac{tdt}{\left(1 + \frac{1}{4}t^2\right)(e^{\pi t} - 1)}$$

and for $\varphi x = \sin ax$, $\int_0^\infty \frac{e^{at}-e^{-at}}{e^{\pi t}-1} dt = \frac{1}{a} - \cot a$.

The second Abel's summation formula is

$$\begin{aligned} &\varphi(x+1) - \varphi(x+2) + \varphi(x+3) - \varphi(x+4) + \dots \quad (25) \\ &= \frac{1}{2}\varphi x + 2\int_0^\infty \frac{dt}{e^{\pi t} - e^{-\pi t}} \frac{\varphi(x + t\sqrt{-1}) - \varphi(x - t\sqrt{-1})}{2\sqrt{-1}}. \end{aligned}$$

In order to obtain this, Abel puts *a priori* the first member equal to $\frac{1}{2}\varphi x + A_1\varphi' x + A_2\varphi'' x + \dots$ with unknown coefficients $A_1, A_2, \dots$; when $\varphi x = e^{cx\sqrt{-1}}$, one sees that $A_2 = A_4 = \dots = 0$ and $\frac{1}{2}\tan\frac{1}{2}c = A_1 c - A_3 c^3 + A_5 c^5 - \dots$. On the other hand, $\frac{1}{2}\tan\frac{1}{2}c = \int_0^\infty \frac{e^{ct}-e^{-ct}}{e^{\pi t}-e^{-\pi t}} dt$ after Legendre (*Exercices de Calcul Intégral*, t. II, p. 186), so the series for $e^{ct} - e^{-ct}$ gives the A_{2n+1} in the form of integrals and the reasoning ends as for the first formula. As an application, Abel takes $\varphi x = \frac{1}{x+1}$ and gets $\int\limits_0^\infty \frac{tdt}{(1+t^2)(e^{\pi t}-e^{-\pi t})} = \frac{1}{2}\log 2 - \frac{1}{4}$.

In the second volume of *Magasinet for Naturvidenskaberne* (1825), Abel published another derivation of the formula (24) and he extended it to the case of iterated sums (*Œuvres*, t. I, p. 34–39):

$$\begin{aligned}\sum^n \varphi x &= A_{n-1,n}\Gamma n \overset{n}{\int} \varphi x \cdot dx^n - A_{n-2,n}\Gamma(n-1) \overset{n-1}{\int} \varphi x \cdot dx^{n-1} \\ &\quad + \ldots + (-1)^{n-1} \int \varphi x \cdot dx + (-1)^n \frac{1}{2}\varphi x \\ &\quad + 2(-1)^{n-1} \int\limits_0^\infty \frac{Pdt}{e^{2\pi t}-1} \frac{\varphi(x+t\sqrt{-1}) - \varphi(x-t\sqrt{-1})}{2\sqrt{-1}} \\ &\quad + 2(-1)^{n-1} \int\limits_0^\infty \frac{Qdt}{e^{2\pi t}-1} \frac{\varphi(x+t\sqrt{-1}) + \varphi(x-t\sqrt{-1})}{2}\end{aligned}$$

where the coefficients $A_{0,n}, A_{1,n}, \ldots, A_{n-1,n}$ are defined by the development of $p^n = \frac{1}{(e^v-1)^n}$ in the form $(-1)^{n-1}\left(A_{0,n}p + A_{1,n}\frac{dp}{dv} + A_{2,n}\frac{d^2p}{dv^2} + \ldots + A_{n-1,n}\frac{d^{n-1}p}{dv^{n-1}}\right)$ and $P = A_{0,n} - A_{2,n}t^2 + A_{4,n}t^4 - \ldots$, $Q = A_{1,n}t - A_{3,n}t^3 + A_{5,n}t^5 - \ldots$; by derivating p^n, Abel establishes recursive relations between the $A_{k,n}$: $A_{0,n+1} - A_{0,n} = 0$, $A_{1,n+1} - A_{1,n} = \frac{1}{n}A_{0,n}$, $A_{2,n+1} - A_{2,n} = \frac{1}{n}A_{1,n}, \ldots, A_{n-1,n+1} - A_{n-1,n} = \frac{1}{n}A_{n-2,n}$, $A_{n,n+1} = \frac{1}{n}A_{n-1,n}$. The proof of this formula is based on the expression of φ as a Laplace transform: $\varphi x = \int e^{vx} fv \cdot dv$, which naturally restricts the generality; it gives

$$\sum^n \varphi x = \int e^{vx} \frac{fv}{(e^v-1)^n} dv.$$

As an example, for $\varphi x = e^{ax}$ and $n = 2$, this formula gives:

$$\frac{1}{(e^a-1)^2} = \frac{1}{2} - \frac{1}{a} + \frac{1}{a^2} - 2\int\limits_0^\infty \frac{dt \cdot \sin at}{e^{2\pi t}-1} - 2\int\limits_0^\infty \frac{tdt \cdot \cos at}{e^{2\pi t}-1}.$$

Another example, with $\varphi x = \frac{1}{x^2}$ and $n = 1$, leads to

$$\frac{1}{a^2} + \frac{1}{(a+1)^2} + \frac{1}{(a+2)^2} + \ldots = \frac{1}{2a^2} + \frac{1}{a} + 4a\int\limits_0^\infty \frac{tdt}{(e^{2\pi t}-1)(a^2+t^2)^2};$$

in particular, for $a = 1$, $\frac{\pi^2}{6} = \frac{3}{2} + 4\int\limits_0^\infty \frac{tdt}{(e^{2\pi t}-1)(1+t^2)^2}$.

A posthumous paper of Abel is devoted to the study of the Laplace transform; its title is *Sur les fonctions génératrices et leurs déterminantes* (*Œuvres*, t. II, p. 67–81, mem. XI) and the study is purely formal. Abel writes an arbitrary function φ of several variables in the form:

$$\varphi(x, y, z, \dots) = \int e^{xu+yv+zp+\dots} f(u, v, p, \dots) du dv dp \dots ,$$

and he calls φ the *generating* function of f and f the *determinant* function of φ, in abbreviation $\varphi = \mathrm{fg} f$ and $f = \mathrm{D}\varphi$. The following properties of the transform are established in the case of one variable only: linearity, effect of a translation $\mathrm{D}\varphi(x+a) = e^{av}\mathrm{D}\varphi x$ and $\mathrm{fg}(e^{av}\mathrm{D}\varphi x) = \varphi(x+a)$, effect of derivations or integrations $\mathrm{D}\left(\frac{d^n\varphi x}{dx^n}\right) = v^n \cdot \mathrm{D}\varphi x$, $\mathrm{fg}(v^n\mathrm{D}\varphi x) = \frac{d^n\varphi x}{dx^n}$, $\mathrm{D}\left(\int^n \varphi x dx^n\right) = v^{-n}\mathrm{D}\varphi x$ and $\mathrm{fg}(v^{-n}\mathrm{D}\varphi x) = \int^n \varphi x dx^n$, effect of finite differences or iterated sums

$$\mathrm{D}\Delta_\alpha^n \varphi x = (e^{v\alpha} - 1)^n fv, \mathrm{fg}((e^{v\alpha} - 1)^n fv) = \Delta_\alpha^n \varphi x, \mathrm{D}\Sigma_\alpha^n(\varphi x) = (e^{va} - 1)^{-n} fv$$
$$\text{and} \quad \mathrm{fg}((e^{v\alpha} - 1)^{-n} fv) = \Sigma_\alpha^n(\varphi x).$$

Abel also states the effect of the composition of a translation, a derivation and a certain number of finite differences. More generally, if the operator δ is defined by

$$\delta(\varphi x) = A_{n,\alpha}\frac{d^n\varphi(x+\alpha)}{dx^n} + A_{n',\alpha'}\frac{d^{n'}\varphi(x+\alpha')}{dx^{n'}} + \dots , \qquad (26)$$

where $A_{n,\alpha}, A_{n',\alpha'}, \dots$ are constant coefficients, one has $\mathrm{D}(\delta\varphi x) = \psi v \cdot \mathrm{D}\varphi x$ where

$$\psi v = A_{n,\alpha} v^n e^{v\alpha} + A_{n',\alpha'} v^{n'} e^{v\alpha'} + \dots ,$$

and Abel considers the composition of an arbitrary number of operators of the type of δ.

Abel clearly understood how the Laplace transform gives a symbolic calculus on the operators (26); he uses this calculus to obtain developments in series. For instance, he explains that the Taylor series for $\varphi(x+\alpha)$ amounts to the development $e^{v\alpha} = 1 + v\alpha + \frac{v^2}{1\cdot 2}\alpha^2 + \frac{v^3}{1\cdot 2\cdot 3}\alpha^3 + \dots$ in the determinant function. A polynomial relation between the multiplicators $\psi, \psi_1, \dots, \psi_\mu$ associated to operators $\delta, \delta_1, \dots, \delta_\mu$ gives an analogous relation between the operators themselves. Let us consider the operator $\delta\varphi x = \varphi(x+\alpha) + a\varphi x$; one has $\mathrm{D}\delta\varphi x = (e^{v\alpha} + a) fv$ where f is the determinant function of φ. Since

$$\begin{aligned}(a + e^{v\alpha})^n &= a^n + na^{n-1}e^{v\alpha} + \frac{n(n-1)}{2}a^{n-2}e^{2v\alpha} + \dots\\ &= e^{nv\alpha} + nae^{(n-1)v\alpha} + \frac{n(n-1)}{2}a^2e^{(n-2)v\alpha} + \dots ,\\ \delta^n\varphi x &= a^n\varphi x + na^{n-1}\varphi(x+\alpha) + \frac{n(n-1)}{2}a^{n-2}\varphi(x+2\alpha) + \dots\\ &= \varphi(x+n\alpha) + na\varphi(x+(n-1)\alpha) + \frac{n(n-1)}{2}a^2\varphi(x+(n-2)\alpha) + \dots ;\end{aligned}$$

Abel writes down both forms, which are the same for n a natural integer, but which may be extended (under different conditions for the convergence of the series) to other values of n; he says nothing about that, but he may have envisaged this type of extension as we saw that he was interested by derivatives of non-integral order and we know that Cauchy defined pseudo-differential operators with constant coefficients using the Fourier transform (1827). For $a = -1$, $\Delta_\alpha^n \varphi x = \varphi(x+n\alpha) - n\varphi(x+(n-1)\alpha) + \frac{n(n-1)}{2}\varphi(x+(n-2)\alpha) - \ldots$

Now let $\delta_1\varphi x = \varphi(x+\alpha_1) + a_1\varphi x$, so that $\mathrm{D}\delta_1^n\varphi x = y^n fv$ with $y = e^{v\alpha_1} + a_1$; if $z = e^{v\alpha} + a$ then $y = a_1 + (z-a)^{\frac{\alpha_1}{\alpha}}$ and it is possible to get a development $y^n = \sum A_m z^m$. Therefore $\delta_1^n\varphi x = \sum A_m \delta^m \varphi x$. In the case where $\alpha_1 = \alpha$,

$$\begin{aligned}\delta_1^n\varphi x &= (a_1-a)^n\varphi x + n(a_1-a)^{n-1}\delta\varphi x + \frac{n(n-1)}{2}(a_1-a)^{n-2}\delta^2\varphi x + \ldots\\ &= \delta^n\varphi x + n(a_1-a)\delta^{n-1}\varphi x + \frac{n(n-1)}{2}(a_1-a)^2\delta^{n-2}\varphi x + \ldots\end{aligned}$$

For $a_1 = 0$, $\varphi(x+n\alpha) = \delta^n\varphi x - na\delta^{n-1}\varphi x + \frac{n(n-1)}{2}a^2\delta^{n-2}\varphi x + \ldots$ and if moreover $a = -1$, $\varphi(x+n\alpha) = \Delta_\alpha^n\varphi x + n\Delta_\alpha^{n-1}\varphi x + \frac{n(n-1)}{2}\Delta_\alpha^{n-2}\varphi x + \ldots$, a formula given by Euler (1755).

When $\delta\varphi x = \varphi(x+\alpha) - a\varphi x$ and $\delta_1\varphi x = c\varphi x + k\frac{d\varphi x}{dx}$, $\mathrm{D}\delta\varphi x = (e^{v\alpha}-a)fv = zfv$ and $\mathrm{D}\delta_1^n\varphi x = (c+kv)^n fv = y^n fv$; as $y = c + \frac{k}{\alpha}\log(z+a) = c + \frac{k}{\alpha}\log a + \frac{k}{\alpha}\left(\frac{z}{a} - \frac{1}{2}\frac{z^2}{a^2} + \frac{1}{3}\frac{z^3}{a^3} - \ldots\right)$, one may write a development $y^n = \sum A_m z^m$, which gives $\delta_1^n\varphi x = \sum A_m\delta^m\varphi x$. For example, if $c = 0$, $a = k = 1$ and $n = 1$, $\frac{d\varphi x}{dx} = \frac{1}{\alpha}\left(\Delta\varphi x - \frac{1}{2}\Delta^2\varphi x + \frac{1}{3}\Delta^3\varphi x - \ldots\right)$, a formula given by Lagrange (1772). Starting from a formula of Legendre:

$$b^v = 1 + lb\cdot vc^v + lb(lb-2lc)\frac{(vc^v)^2}{2} + lb(lb-3lc)^2\frac{(vc^v)^3}{2\cdot 3} + \ldots,$$

in which he makes $b = e^\alpha$ and $c = e^\beta$, Abel obtains in the same way

$$\begin{aligned}\varphi(x+\alpha) = \varphi x &+ \alpha\frac{d\varphi(x+\beta)}{dx} + \frac{\alpha(\alpha-2\beta)}{2}\cdot\frac{d^2\varphi(x+2\beta)}{dx^2}\\ &+ \frac{\alpha(\alpha-3\beta)^2}{2\cdot 3}\cdot\frac{d^3\varphi(x+3\beta)}{dx^3} + \ldots\end{aligned} \tag{27}$$

and, in particular, $\varphi x = \varphi(0) + x\varphi'(\beta) + \frac{x(x-2\beta)}{2}\varphi''(2\beta) + \frac{x(x-3\beta)^2}{2\cdot 3}\varphi'''(3\beta) + \ldots$ Abel published the special case of (27) in which $\varphi x = x^m$, m a natural integer, in the first issue of *Crelle's Journal* (*Œuvres*, t. I, p. 102–103); there, he proves the formula by induction on m and he observes that, when $\beta = 0$, the result reduces to the binomial formula. Another special case given in the posthumous memoir is that in which $\varphi x = \log x$; then

$$\log(x+\alpha) = \log x + \frac{\alpha}{x+\beta} + \frac{1}{2}\cdot\frac{\alpha}{x+2\beta}\cdot\frac{2\beta-\alpha}{x+2\beta} + \frac{1}{3}\cdot\frac{\alpha}{x+3\beta}\cdot\left(\frac{3\beta-\alpha}{x+3\beta}\right)^2 + \ldots$$

and in particular $\log(1+\alpha) = \frac{\alpha}{1+\beta} + \frac{1}{2}\cdot\frac{\alpha}{1+2\beta}\cdot\left(1-\frac{1+\alpha}{1+3\beta}\right) + \frac{1}{3}\cdot\frac{\alpha}{1+3\beta}\cdot\left(1-\frac{1+\alpha}{1+3\beta}\right)^2 + \dots$ When $\alpha = 3\beta$, this reduces to $\log(1+2\beta) = \frac{2\beta}{1+\beta} + \frac{2}{3}\cdot\frac{\beta^3}{(1+3\beta)^3} + \frac{1}{4}\cdot\frac{2\cdot 2^3\cdot\beta^4}{(1+4\beta)^4} + \dots$; for example, $\log 3 = 1 + \sum_{n\geq 3} \frac{2}{n}\frac{1}{n+1}\left(\frac{n-2}{n+1}\right)^{n-1}$.

Abel also considers the developments of $\Delta_\alpha^n \varphi x$, $\frac{d^n\varphi x}{dx^n}$ and $\frac{d^n(e^x\varphi x)}{dx^n}$ in power series with respect to n; they are respectively obtained from the developments of

$$(e^{v\alpha}-1)^n = \exp(n\log(e^{v\alpha}-1)), \quad v^n = e^{n\log v}, \quad \text{and} \quad (1+v)^n = e^{n\log(1+v)}.$$

The coefficients respectively contain the powers of $\log(e^{v\alpha}-1)$, $\log v$ and $\log(1+v)$, so we must identify the operators δ respectively defined by

$$\delta\varphi x = \mathrm{fg}(\log(e^{v\alpha}-1)\,fv), \quad \log v\cdot fv \quad \text{and} \quad (1+v)\,fv;$$

these operators are respectively $\delta\varphi x = \alpha\varphi' x + \int d\alpha \sum_\alpha \varphi' x$, $\delta_1\varphi x - \frac{1}{2}\delta_1^2\varphi x + \frac{1}{3}\delta_1^3\varphi x - \dots$, where $\delta_1\varphi x = \varphi' x - \varphi x$, and $\varphi' x - \frac{1}{2}\varphi'' x + \frac{1}{3}\varphi''' x - \dots$

In the continuation of the paper, Abel expresses this last operator in the integral form

$$\delta\varphi x = \int_\infty^0 \frac{e^{-t}dt}{t}(\varphi(x-t)-\varphi x),$$

which is obtained in the following manner: the equality

$$\begin{aligned}\int_a^{a'} e^{(1-\alpha v)t}dt &= (e^a e^{-a\alpha v} - e^{a'}e^{-a'\alpha v})\frac{1}{1-\alpha v}\\ &= e^a(e^{-a\alpha v} + \alpha v e^{-a\alpha v} + \alpha^2 v^2 e^{-a\alpha v} + \dots)\\ &\quad - e^{a'}(e^{-a'\alpha v} + \alpha v e^{-a'\alpha v} + \alpha^2 v^2 e^{-a'\alpha v} + \dots)\end{aligned}$$

leads to

$$\begin{aligned}&\int_a^{a'} e^t\varphi(x-\alpha t)dt =\\ &e^a(\varphi(x-\alpha a) + \alpha\varphi'(x-\alpha a) + \alpha^2\varphi''(x-\alpha a) + \alpha^3\varphi'''(x-\alpha a) + \dots)\\ &-e^{a'}(\varphi(x-\alpha a') + \alpha\varphi'(x-\alpha a') + \alpha^2\varphi''(x-\alpha a') + \alpha^3\varphi'''(x-\alpha a') + \dots),\end{aligned}$$

from which Abel deduces $\varphi' x - \alpha\varphi'' x + \alpha^2\varphi''' x - \alpha^3\varphi'''' x + \dots = \int_{-\infty}^0 e^t\varphi'(x+\alpha t)dt$ and, integrating with respect to α, $\alpha\varphi' x - \frac{1}{2}\alpha^2\varphi'' x + \frac{1}{3}\alpha^3\varphi''' x - \frac{1}{4}\alpha^4\varphi'''' x + \dots = \int_{-\infty}^0 \frac{e^t dt}{t}(\varphi(x+\alpha t) - \varphi x)$.

Other classical relations between functions give Abel relations between operators. So the Fourier series $\frac{1}{2} = \cos\alpha v - \cos 2\alpha v + \cos 3\alpha v - \dots$ leads to

$$\varphi x = \varphi(x+\alpha) + \varphi(x-\alpha) - \varphi(x+2\alpha) - \varphi(x-2\alpha) + \varphi(x+3\alpha) + \varphi(x-3\alpha) \\ -\varphi(x+4\alpha) - \varphi(x-4\alpha) + \dots$$

and the formula $(e^{v\alpha}-1)^{-1} - (v\alpha)^{-1} + \frac{1}{2} = 2\int\limits_0^\infty \frac{dt\cdot\sin(v\alpha t)}{e^{2\pi t}-1}$ leads to

$$\sum\nolimits_\alpha \varphi x - \frac{1}{\alpha}\int \varphi x dx + \frac{1}{2}\varphi x = 2\int\limits_0^\infty \frac{dt}{e^{2\pi t}-1}\cdot\frac{\varphi(x+\alpha t\sqrt{-1}) - \varphi(x-\alpha t\sqrt{-1})}{2\sqrt{-1}},$$

which is formula (24). From the formula $\int\limits_0^\infty \frac{dt\cdot\cos(\alpha v t)}{1+t^2} = \frac{\pi}{2}e^{-\alpha v}$ given by Legendre (*Exercices de Calcul intégral*, t. II, p. 176), Abel deduces

$$\int\limits_0^\infty \frac{dt}{1+t^2}\cdot\frac{\varphi(x+\alpha t\sqrt{-1}) + \varphi(x-\alpha t\sqrt{-1})}{2} = \frac{\pi}{2}\varphi(x\pm\alpha), \tag{28}$$

for instance $\int\limits_0^\infty \frac{dt}{(1+t^2)(\alpha^2t^2+x^2)} = \frac{\pi}{2}\cdot\frac{1}{x(x\pm\alpha)}$ for $\varphi x = \frac{1}{x}$ (where it is easy to verify that $\pm$ must be taken as $+$); when $\varphi x = \frac{1}{x^n}$, $\frac{\varphi(x+\alpha t\sqrt{-1})+\varphi(x-\alpha t\sqrt{-1})}{2} = z^{-n}\cos n\phi$, where $z = \sqrt{x^2+\alpha^2t^2}$ and $\phi = \arctan\frac{\alpha t}{x}$, so that $\int\limits_0^\infty \frac{dt}{1+t^2}\cdot\frac{\cos\left(n\arctan\frac{\alpha t}{x}\right)}{(x^2+\alpha^2t^2)^{\frac{n}{2}}} = \frac{\pi}{2}\cdot\frac{1}{(x+\alpha)^n}$ or

$$\frac{\pi}{2}\cdot\frac{x^{n-1}}{\alpha(x+\alpha)^n} = \int\limits_0^{\frac{\pi}{2}} \frac{(\cos\phi)^n\cos n\phi d\phi}{(x\sin\phi)^2 + (\alpha\cos\phi)^2}, \tag{29}$$

which reduces to

$$\frac{\pi}{2^{n+1}} = \int\limits_0^{\frac{\pi}{2}} (\cos\phi)^n \cos n\phi d\phi \tag{30}$$

when $\alpha = x$.

From the integrals $\int\limits_0^\infty \frac{dt\cdot\sin at}{t(1+t^2)} = \frac{\pi}{2}(1-e^{-a})$ and $\int\limits_0^\infty \frac{tdt\cdot\sin at}{1+t^2} = \frac{\pi}{2}e^{-a}$ also given by Legendre, Abel deduces $\frac{\pi}{2}(\varphi x - \varphi(x\pm\alpha)) = \int\limits_0^\infty \frac{dt}{t(1+t^2)}\cdot\frac{\varphi(x+\alpha t\sqrt{-1})-\varphi(x-\alpha t\sqrt{-1})}{2\sqrt{-1}}$, $\frac{\pi}{2}\varphi(x\pm\alpha) = \int\limits_0^\infty \frac{tdt}{1+t^2}\cdot\frac{\varphi(x+\alpha t\sqrt{-1})-\varphi(x-\alpha t\sqrt{-1})}{2\sqrt{-1}}$ and $\frac{\pi}{2}\varphi x = \int\limits_0^\infty \frac{dt}{t}\cdot\frac{\varphi(x+\alpha t\sqrt{-1})-\varphi(x-\alpha t\sqrt{-1})}{2\sqrt{-1}}$; for $\varphi x = \frac{1}{x^n}$ this gives $\int\limits_0^{\frac{\pi}{2}} \frac{d\phi}{\sin\phi}(\cos\phi)^{n-1}\sin n\phi = \frac{\pi}{2}$ by putting $t = x\tan\phi$.

In an addition to this paper (*Sur quelques intégrales définies*, *Œuvres*, t. II, p. 82–86, mem. XII), Abel develops $(\cos\phi)^n\cos n\phi$ and $\frac{x^n}{(x+\alpha)^n}$ in power series with respect to n and, comparing the coefficients of the powers of n in (29), he gets the values of some definite integrals:

$$\frac{\pi}{2}\cdot\frac{1}{x\alpha}=\int_0^{\frac{\pi}{2}}\frac{d\phi}{x^2\sin^2\phi+\alpha^2\cos^2\phi},\ \frac{\pi}{2}\cdot\frac{1}{x\alpha}\log\frac{x}{x+\alpha}=\int_0^{\frac{\pi}{2}}\frac{\log\cos\phi d\phi}{x^2\sin^2\phi+\alpha^2\cos^2\phi},$$

$$\frac{\pi}{2}\cdot\frac{1}{x\alpha}\left(\log\frac{x}{x+\alpha}\right)^2=\int_0^{\frac{\pi}{2}}\frac{((\log\cos\phi)^2-\phi^2)d\phi}{x^2\sin^2\phi+\alpha^2\cos^2\phi}.$$

Putting $\varphi x=(\log x)^n$ and $\frac{\alpha t}{x}=\tan\phi$ in (28), he gets

$$\int_0^{\frac{\pi}{2}}\frac{d\phi}{x^2\sin^2\phi+\alpha^2\cos^2\phi}\cdot\frac{\left(\log\frac{x}{\cos\phi}+\phi\sqrt{-1}\right)^n+\left(\log\frac{x}{\cos\phi}-\phi\sqrt{-1}\right)^n}{2}$$
$$=\frac{\pi}{2x\alpha}(\log(x+\alpha))^n$$

and $\int_0^{\frac{\pi}{2}}d\phi\left(\left(\log\frac{x}{\cos\phi}+\phi\sqrt{-1}\right)^n+\left(\log\frac{x}{\cos\phi}-\phi\sqrt{-1}\right)^n\right)=\pi(\log 2)^n$ when $x=\alpha=1$.

More generally, putting $t=\tan u$ in (28) we get

$$\int_0^{\frac{\pi}{2}}du(\varphi(x+\alpha\sqrt{-1}\tan u)+\varphi(x-\alpha\sqrt{-1}\tan u))$$

$=\pi\varphi(x+\alpha)$ and $\int_0^{\frac{\pi}{2}}du(\varphi(1+\sqrt{-1}\tan u)+\varphi(1-\sqrt{-1}\tan u))=\pi\varphi(2)$ when $x=\alpha=1$; for $\varphi x=\frac{x^m}{1+\alpha x^n}$, this gives

$$\int_0^{\frac{\pi}{2}}\frac{(\cos u)^{n-m}(\cos mu(\cos u)^n+\alpha\cos(n-m)u)}{(\cos u)^{2n}+2\alpha\cos nu(\cos u)^n+\alpha^2}du=\frac{\pi}{2}\cdot\frac{2^m}{1+\alpha 2^n}.$$

In (30) Abel replaces n by a fraction $\frac{m}{n}$ and he puts $\frac{\phi}{n}=\theta$, so that

$$\frac{\pi}{2n}\cdot\frac{1}{2^{\frac{m}{n}}}=\int_0^{\frac{\pi}{2}}(\cos n\theta)^{\frac{m}{n}}\cos m\theta d\theta=-\int_1^{\cos\frac{\pi}{2n}}\sqrt[n]{(\psi y)^m}\,fy\frac{dy}{\sqrt{1-y^2}},$$

where $\psi y = y^n - \frac{n(n-1)}{2} y^{n-2}(1-y^2) + \frac{n(n-1)(n-2)(n-3)}{2\cdot3\cdot4} y^{n-4}(1-y^2)^2 - \dots$ and $fy = y^m - \frac{m(m-1)}{2} y^{m-2}(1-y^2) + \frac{m(m-1)(m-2)(m-3)}{2\cdot3\cdot4} y^{m-4}(1-y^2)^2 - \dots$; for instance $\frac{\pi}{8}\cdot\frac{1}{\sqrt[4]{2}} = -\int_1^{\cos\frac{\pi}{8}} \sqrt[4]{1-8y^2+8y^4}\frac{ydy}{\sqrt{1-y^2}} = \int_0^{\sin\frac{\pi}{8}} dz\sqrt[4]{1-8z^2+8z^4}$.

All this early work of Abel gives evidence of his carefull study of Legendre's *Exercices de Calcul intégral*, which also were his source of inspiration for the theory of elliptic integrals.

A paper on a related subject was published by Abel in the second volume of *Crelle's Journal* (1827, *Œuvres*, t. I, p. 251–262) under the title *Sur quelques intégrales définies*. It contains some applications of the relation discovered by Abel $y_2\frac{dy_1}{da} - y_1\frac{dy_2}{da} = e^{-\int pda}$, where y_1 and y_2 are two solutions of the linear differential equation $\frac{d^2u}{da^2} + p\frac{dy}{da} + qy = 0$. For instance $y_1 = \int_0^1 \frac{(x+a)^{\gamma+1}dx}{x^{1-\alpha}(1-x)^{1-\beta}}$ and $y_2 = \int_0^1 \frac{(x+a)^{\alpha+\beta+\gamma}dx}{x^{\beta}(1-x)^{\alpha}}$ are solutions of the hypergeometric equation

$$\frac{d^2y}{da^2} - \left(\frac{\alpha+\gamma}{a} + \frac{\beta+\gamma}{1+a}\right)\frac{dy}{da} + \frac{(\gamma+1)(\alpha+\beta+\gamma)}{a(a+1)}y = 0,$$

and this leads to the relation

$$\begin{aligned}&(\alpha+\beta+\gamma)\int_0^1 \frac{dx(x+a)^{\gamma+1}}{x^{1-\alpha}(1-x)^{1-\beta}}\cdot\int_0^1 \frac{dx(x+a)^{\alpha+\beta+\gamma-1}}{x^{\beta}(1-x)^{\alpha}}\\ &-(\gamma+1)\int_0^1 \frac{dx(x+a)^{\gamma}}{x^{1-\alpha}(1-x)^{1-\beta}}\cdot\int_0^1 \frac{dx(x+a)^{\alpha+\beta+\gamma}}{x^{\beta}(1-x)^{\alpha}} = Ca^{\alpha+\gamma}(1+a)^{\beta+\gamma},\end{aligned}$$

where the constant C is determined by making $a = \infty$:

$$\begin{aligned}C &= -(\alpha+\beta-1)\int_0^1 dx\cdot x^{\alpha-1}(1-x)^{\beta-1}\cdot\int_0^1 dx\cdot x^{-\beta}(1-x)^{-\alpha}\\ &= \pi(\cot\alpha\pi + \cot\beta\pi).\end{aligned}$$

In the same way $y_1 = \int_0^\infty \frac{x^{-\alpha}dx}{(1+x)^{\beta}(x+a)^{\gamma}}$ and $y_2 = \int_0^\infty \frac{x^{\beta-1}dx}{(1+x)^{1-\alpha}(x+a)^{\alpha+\beta+\gamma-1}}$ are solutions of the hypergeometric equation $\frac{d^2y}{da^2} + \left(\frac{\alpha+\gamma}{a} - \frac{\beta+\gamma}{1-a}\right)\frac{dy}{da} + \frac{\gamma(1-\alpha-\beta-\gamma)}{a(1-a)}y = 0$ and we get the relation $\int_0^\infty \frac{x^{-\alpha}dx}{(1+x)^{\beta}(x+a)^{\gamma}} = \frac{\Gamma(1-\alpha)\Gamma(\alpha+\beta+\gamma-1)}{\Gamma\beta\cdot\Gamma\gamma}\int_0^\infty \frac{x^{\beta-1}dx}{(1+x)^{1-\alpha}(x+a)^{\alpha+\beta+\gamma-1}}$. The function $y_3 = \int_0^\infty \frac{x^{-\beta}dx}{(1+x)^{\alpha}(x+1-a)^{\gamma}} = (1-a)^{-\beta-\gamma+1}\int_0^\infty \frac{x^{-\beta}dx}{(1+x)^{\gamma}(1+(1-a)x)^{\alpha}}$ is a third solution of the same hypergeometric equation, which, combined with $y_1 = a^{-\alpha-\gamma+1}\int\limits_0^\infty \frac{x^{-\alpha}dx}{(1+x)^\gamma(1+ax)^\beta}$, gives

$$a\int\limits_0^\infty \frac{x^{-\alpha}dx}{(1+x)^\gamma(1+ax)^\beta}\cdot\int\limits_0^\infty \frac{x^{-\beta}dx}{(1+x)^{\gamma+1}(1+(1-a)x)^\alpha}$$
$$+(1-a)\int\limits_0^\infty \frac{x^{-\beta}dx}{(1+x)^\gamma(1+(1-a)x)^\alpha}\cdot\int\limits_0^\infty \frac{x^{-\alpha}dx}{(1+x)^{\gamma+1}(1+ax)^\beta}$$
$$=\frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(\gamma+1)}\Gamma(\alpha+\beta+\gamma-1).$$

When $\beta = 1-\alpha$, this relation becomes $a\int\limits_0^\infty \frac{x^{-\alpha}dx}{(1+x)^\gamma(1+ax)^{1-\alpha}}\cdot\int\limits_0^\infty \frac{x^{\alpha-1}dx}{(1+x)^{\gamma+1}(1+(1-a)x)^\alpha}+(1-a)\int\limits_0^\infty \frac{x^{-\alpha}dx}{(1+x)^{\gamma+1}(1+ax)^{1-\alpha}}\cdot\int\limits_0^\infty \frac{x^{\alpha-1}dx}{(1+x)^\gamma(1+(1-a)x)^\alpha} = \frac{\pi}{\gamma\cdot\sin\alpha\pi}$; in particular, for $\alpha=\gamma=\frac{1}{2}$,

$$a\int\limits_0^\infty \frac{dx}{\sqrt{x(1+x)(1+ax)}}\cdot\int\limits_0^\infty \frac{dx}{\sqrt{x(1+x)^3(1+(1-a)x)}}$$
$$+(1-a)\int\limits_0^\infty \frac{dx}{\sqrt{x(1+x)(1+(1-a)x)}}\cdot\int\limits_0^\infty \frac{dx}{\sqrt{x(1+x)^3(1+ax)}}=2\pi.$$

As Abel observes, these integrals are elliptic and the change of variable $x=\tan^2\varphi$ transforms the preceding relation in

$$a\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-(1-a)\sin^2\varphi}}\cdot\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi\cdot\cos^2\varphi}{\sqrt{1-a\sin^2\varphi}}$$
$$+(1-a)\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-a\sin^2\varphi}}\cdot\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi\cdot\cos^2\varphi}{\sqrt{1-(1-a)\sin^2\varphi}}=\frac{\pi}{2},$$

which is equivalent to Legendre's famous relation between the complete integrals of the first two kinds (*Exercices de Calcul intégral*, t. I, p. 61). Legendre had proved this relation by a very similar method.

Starting from the integral $y=\int\limits_0^x \frac{dx\cdot x^{\alpha-1}(1-x)^{\beta-1}}{(x+a)^{\alpha+\beta}}$, Abel finds that $\frac{dy}{da}+\left(\frac{\alpha}{1+a}+\frac{\beta}{a}\right)y = -\frac{x^\alpha(1-x)^\beta}{a(1+a)(x+a)^{\alpha+\beta}}$, so that $y\cdot a^\beta(1+a)^\alpha = C - x^\alpha(1-x)^\beta\int\limits_0^a \frac{da\cdot a^{\beta-1}(1+a)^{\alpha-1}}{(a+x)^{\alpha+\beta}}$, where C is independent of a and is found to be $\frac{\Gamma\alpha\cdot\Gamma\beta}{\Gamma(\alpha+\beta)}$ by making $a=\infty$. Thus

$$\frac{\Gamma\alpha\cdot\Gamma\beta}{\Gamma(\alpha+\beta)}=a^{\beta}(1+a)^{\alpha}\int_0^x\frac{dx\cdot x^{\alpha-1}(1-x)^{\beta-1}}{(x+a)^{\alpha+\beta}}$$
$$+x^{\alpha}(1-x)^{\beta}\int_0^a\frac{da\cdot a^{\beta-1}(1+a)^{\alpha-1}}{(a+x)^{\alpha+\beta}};$$

when $\alpha+\beta=1$, this gives

$$\frac{(1+a)^{\alpha}}{a^{\alpha-1}}\int_0^x\frac{dx\cdot x^{\alpha-1}(1-x)^{-\alpha}}{x+a}+\frac{x^{\alpha}}{(1-x)^{\alpha-1}}\int_0^a\frac{da\cdot a^{-\alpha}(1+a)^{\alpha-1}}{a+x}=\frac{\pi}{\sin\pi\alpha}.$$

The integral $y=\int_0^1 e^{-ax}x^{\alpha-1}(1-x)^{\beta-1}dx$ ($\alpha,\beta>0$) is a solution of the confluent hypergeometric equation $\frac{d^2y}{da^2}+\left(\frac{\alpha+\beta}{a}+1\right)\frac{dy}{da}+\frac{\alpha}{a}y=0$, and so is

$$y_1=\int_1^{\infty}e^{-ax}x^{\alpha-1}(1-x)^{\beta-1}dx$$
$$=e^{-a}\int_0^{\infty}e^{-ax}x^{\beta-1}(1+x)^{\alpha-1}dx=e^{-a}a^{-\alpha-\beta+1}\int_0^{\infty}e^{-x}x^{\beta-1}(a+x)^{\alpha-1}dx$$

($a>0$). Abel derives from that the formula

$$\Gamma\alpha\cdot\Gamma\beta=\int_0^1 e^{-ax}x^{\alpha-1}(1-x)^{\beta-1}dx\cdot\int_0^{\infty}e^{-x}x^{\beta-1}(a+x)^{\alpha}dx$$
$$-a\int_0^1 e^{-ax}x^{\alpha}(1-x)^{\beta-1}dx\cdot\int_0^{\infty}e^{-x}x^{\beta-1}(a+x)^{\alpha-1}dx,$$

and, for $\beta=1-\alpha$,

$$\frac{\pi}{\sin\pi\alpha}=\int_0^1\frac{dx}{x}e^{-ax}\left(\frac{x}{1-x}\right)^{\alpha}\cdot\int_0^{\infty}e^{-x}dx\left(1+\frac{a}{x}\right)^{\alpha}$$
$$-a\int_0^1 dx\cdot e^{-ax}\left(\frac{x}{1-x}\right)^{\alpha}\cdot\int_0^{\infty}\frac{dx}{x+a}e^{-x}\left(1+\frac{a}{x}\right)^{\alpha}.$$

As a last example, Abel considers the integrals $y=\int_0^{\infty}e^{ax-x^2}x^{\alpha-1}dx$ and $y_1=\int_0^{\infty}e^{-ax-x^2}x^{\alpha-1}dx$ ($\alpha>0$), solutions of the differential equation $\frac{d^2y}{da^2}-\frac{1}{2}a\frac{dy}{da}-\frac{1}{2}\alpha y=0$, which is related to the so called Weber equation. The corresponding relation is

$$\frac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)e^{\frac{a^2}{4}} = \int_0^\infty e^{ax-x^2}x^{\alpha-1}dx \cdot \int_0^\infty e^{-ax-x^2}x^{\alpha}dx + \int_0^\infty e^{ax-x^2}x^{\alpha}dx \cdot \int_0^\infty e^{-ax-x^2}x^{\alpha-1}dx.$$

In a posthumous paper *Les fonctions transcendantes* $\sum\frac{1}{a^2}, \sum\frac{1}{a^3}, \sum\frac{1}{a^4}, \ldots, \sum\frac{1}{a^n}$ *exprimées par des intégrales définies* (*Œuvres*, t. II, p. 1–6), Abel gives integral formulae for these finite sums, extended from 1 to $a-1$. He also studies their continuation to non integral values of a and n. As $\frac{d^n\sum\frac{1}{a}}{da^n} = (-1)^n 2\cdot 3\cdots n\sum\frac{1}{a^{n+1}}$, one has

$$\sum\frac{1}{a^n} = (-1)^{n-1}\frac{d^{n-1}L(a)}{2\cdot 3\cdots(n-1)da^{n-1}}$$

where $L(a) = \sum\frac{1}{a} = \int_0^1\frac{x^{a-1}-1}{x-1}dx$. From this Abel deduces

$$L(a,\alpha) = \sum\frac{1}{a^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^1\frac{x^{a-1}-1}{x-1}\left(l\frac{1}{x}\right)^{\alpha-1}dx \tag{31}$$

for any value of α. Substituting $x^{a-1} = 1-(a-1)\left(l\frac{1}{x}\right)+\frac{(a-1)^2}{2}\left(l\frac{1}{x}\right)^2-\frac{(a-1)^3}{2\cdot 3}\left(l\frac{1}{x}\right)^3+\ldots$ and $\frac{1}{1-x} = 1+x+x^2+\ldots$, he obtains

$$\begin{aligned}L(a,\alpha) &= \frac{a-1}{1}\alpha\left(1+\frac{1}{2^{\alpha+1}}+\frac{1}{3^{\alpha+1}}+\frac{1}{4^{\alpha+1}}+\ldots\right)\\ &\quad -\frac{(a-1)^2}{1\cdot 2}\alpha(\alpha+1)\left(1+\frac{1}{2^{\alpha+2}}+\frac{1}{3^{\alpha+2}}+\frac{1}{4^{\alpha+2}}+\ldots\right)\\ &\quad +\frac{(a-1)^3}{1\cdot 2\cdot 3}\alpha(\alpha+1)(\alpha+2)\left(1+\frac{1}{2^{\alpha+3}}+\frac{1}{3^{\alpha+3}}+\frac{1}{4^{\alpha+3}}+\ldots\right)\\ &= \alpha(a-1)L'(\alpha+1)-\frac{\alpha(\alpha+1)}{2}(a-1)^2L'(\alpha+2)\\ &\quad +\frac{\alpha(\alpha+1)(\alpha+2)}{2\cdot 3}(a-1)^3L'(\alpha+3)-\ldots,\end{aligned}$$

where $L'(\alpha) = L(\infty,\alpha) = 1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\frac{1}{4^\alpha}+\ldots$ is the zeta function. Putting $\frac{m}{a}$ instead of a in (31), Abel deduces $L\left(\frac{m}{a},\alpha\right) = -\frac{1}{\Gamma(\alpha)}\int_0^1\frac{\left(l\frac{1}{y}\right)^{\alpha-1}}{y-1}dy + \frac{a^\alpha}{\Gamma(\alpha)}\int_0^1\frac{y^{m-1}\left(l\frac{1}{y}\right)^{\alpha-1}}{y^a-1}dy$ or, writing $\frac{y^{m-1}}{y^a-1} = \frac{A}{1-cy}+\frac{A'}{1-c'y}+\ldots$ in the hypothesis $m-1<a$,

$$L\left(\frac{m}{a},\alpha\right) = a^{\alpha}(AL'(\alpha,c) + A'L'(\alpha,c') + A''L'(\alpha,c'') + \dots)$$

where $L'(\alpha,c) = 1 + \frac{c}{2^{\alpha}} + \frac{c^2}{3^{\alpha}} + \frac{c^3}{4^{\alpha}} + \dots$

The following paper *Sur l'intégrale définie* $\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^{\alpha-1}dx$ (*Œuvres*, t. II, p. 7–13) is related to the same subject; it gives developments in series for La and associated functions. When $\alpha = 1$, the integral is equal to $\frac{\Gamma a\cdot\Gamma c}{\Gamma(a+c)}$. As the logarithmic derivative of Γa is equal to $La - C$, where C is the Euler constant, Abel deduces from this $\int_0^1 x^{a-1}(1-x)^{c-1}lxdx = (La - L(a+c))\frac{\Gamma a\cdot\Gamma c}{\Gamma(a+c)}$, $\int_0^1 x^{a-1}(1-x)^{c-1}l(1-x)dx = (Lc - L(a+c))\frac{\Gamma a\cdot\Gamma c}{\Gamma(a+c)}$. For $c = 1$, this gives $\int_0^1 x^{a-1}lxdx = -\frac{1}{a^2}$, $\int_0^1 x^{a-1}l(1-x)dx = -\frac{L(1+a)}{a}$. Developing $(1-x)^{c-1}$ in series, Abel obtains

$$\begin{aligned}(La - L(a+c))\frac{\Gamma a\cdot\Gamma c}{\Gamma(a+c)} &= \frac{1}{a^2} - (c-1)\frac{1}{(a+1)^2} + \frac{(c-1)(c-2)}{2}\frac{1}{(a+2)^2}\\ &\quad - \frac{(c-1)(c-2)(c-3)}{2\cdot 3}\frac{1}{(a+3)^2} + \dots \qquad (32)\end{aligned}$$

For example, if $c = 1-a$, $-La\cdot\frac{\pi}{\sin a\pi} = \frac{1}{a^2} + \frac{a}{(a+1)^2} + \frac{a(a+1)}{2(a+2)^2} + \frac{a(a+1)(a+2)}{2\cdot 3(a+3)^2} + \dots$, which becomes $2\pi\log 2 = 2^2 + \frac{2}{3^2} + \frac{3}{2\cdot 5^2} + \frac{3\cdot 5}{2^2\cdot 3\cdot 7^2} + \frac{3\cdot 5\cdot 7}{2^3\cdot 3\cdot 4\cdot 9^2} + \dots$ when $a = \frac{1}{2}$ for $L\frac{1}{2} = -2\log 2$. When $a = 1-x$ and $c = 2x-1$, $L(1-x) - Lx = \pi\cot\pi x$ and

$$\begin{aligned}-\pi\cot\pi x\frac{\Gamma(1-x)\Gamma(2x-1)}{\Gamma x} &= \frac{1}{(1-x)^2} - \frac{2x-2}{(2-x)^2} + \frac{(2x-2)(2x-3)}{2(3-x)^2}\\ &\quad - \frac{(2x-2)(2x-3)(2x-4)}{2\cdot 3(4-x)^2} + \dots\end{aligned}$$

From (32) Abel deduces an expression of $\frac{L(a+c)-La}{L(a+c)-Lc}$ as a quotient of two series and, making $c = 1$, $L(1+a) = a - \frac{a(a-1)}{2^2} + \frac{a(a-1)(a-2)}{2\cdot 3^2} - \dots$. Thus

$$\begin{aligned}\pi\cot\pi a &= L(1-a) - La\\ &= -\left(2a - 1 + \frac{a(a+1) - (a-1)(a-2)}{2^2}\right.\\ &\qquad \left. + \frac{a(a+1)(a+2) - (a-1)(a-2)(a-3)}{2\cdot 3^2} + \dots\right).\end{aligned}$$

The integral of the title, with α an integer, is obtained by successive differentiations with respect to a:

$$\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^{\alpha-1}dx = \Gamma\alpha\left(\frac{1}{a^{\alpha}} - \frac{c-1}{1}\frac{1}{(a+1)^{\alpha}} + \frac{(c-1)(c-2)}{1\cdot 2}\frac{1}{(a+2)^{\alpha}} - \dots\right).$$

Taking the successive logarithmic derivatives, Abel sees that this integral has an expression in terms of the sums La, $L'a = \sum \frac{1}{a^2}$, $L''a = \sum \frac{1}{a^3}, \ldots$; for example

$$\int_0^1 x^{a-1}(1-x)^{c-1}\left(l\frac{1}{x}\right)^3 dx = \Big(2(L''(a+c) - L''a) \\ +3(L'(a+c) - L'a)(L(a+c) - La) \\ +(L(a+c) - La)^3\Big)\frac{\Gamma a \cdot \Gamma c}{\Gamma(a+c)}.$$

The successive differentiations of the equality $\int_0^1 \left(l\frac{1}{x}\right)^{\alpha-1} dx = \Gamma\alpha$ with respect to α give the formula $\int_0^1 \left(l\frac{1}{x}\right)^{\alpha-1}\left(ll\frac{1}{x}\right)^n dx = \frac{d^n \Gamma\alpha}{d\alpha^n}$, whence $\int_0^\infty (lz)^n e^{-z^{\frac{1}{\alpha}}} dz = \alpha^{n+1}\frac{d^n\Gamma\alpha}{d\alpha^n}$ by a change of variable. Abel deduces from this the formulae $\int_0^\infty e^{-x^\alpha} dx = \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)$ $(n = 0)$ and $\int_0^\infty l\left(\frac{1}{x}\right) e^{-x^\alpha} dx = -\frac{1}{\alpha^2}\Gamma\left(\frac{1}{\alpha}\right)\left(L\left(\frac{1}{\alpha}\right) - C\right)$ ($n = 1$; C is the Euler constant), which leads to

$$\int_0^\infty e^{-nx}x^{\alpha-1}lx dx = \frac{\Gamma\alpha}{n^\alpha}(L\alpha - C - \log n).$$

A third posthumous paper is titled *Sommation de la série* $y = \varphi(0) + \varphi(1)x + \varphi(2)x^2 + \varphi(3)x^3 + \ldots + \varphi(n)x^n$*, n étant un entier positif fini ou infini, et* $\varphi(n)$ *une fonction algébrique rationnelle de n* (*Œuvres*, t. II, p. 14–18). Abel decomposes φ in terms of one of the forms An^α, $\frac{B}{(a+n)^\beta}$. He has first to sum $f(\alpha, x) = x + 2^\alpha x^2 + 3^\alpha x^3 + \ldots + n^\alpha x^n$; this is done using the identities $f(\alpha, x) = \frac{x d f(\alpha-1, x)}{dx}$ and $f(0, x) = \frac{x(1-x^n)}{1-x}$. Then Abel considers

$$F\alpha = \frac{1}{a^\alpha} + \frac{x}{(a+1)^\alpha} + \frac{x^2}{(a+2)^\alpha} + \ldots + \frac{x^n}{(a+n)^\alpha} = \frac{\int dx \cdot x^{\alpha-1}F(\alpha-1)}{x^\alpha},$$

for which $F(0) = \frac{1-x^{n+1}}{1-x}$. The formula (10) for the dilogarithm is thus obtained when $\alpha = 2$, $n = \infty$ and $a = 1$.

3 Algebraic Equations

We know that in 1821 Abel thought he had found a method to solve the general quintic equation by radicals; when he discovered his error and proved that such a solution was impossible, he wrote a booklet in french with a demonstration, *Mémoire sur les* *équations algébriques, où l'on démontre l'impossibilité de la résolution de l'équation générale du cinquième degré* (Christiania, 1824; *Œuvres*, t. I, p. 28–33).

The impossibility of an algebraic solution for the general quintic equation had already been published by P. Ruffini (1799, 1802, 1813), but his demonstration was incomplete for he supposed without proof that the radicals in a hypothetical solution were necessarily rational functions of the roots. Abel, who did not know of Ruffini's work, began with a proof of this fact.

Supposing the root of

$$y^5 - ay^4 + by^3 - cy^2 + dy - e = 0 \tag{33}$$

of the form

$$y = p + p_1 R^{\frac{1}{m}} + p_2 R^{\frac{2}{m}} + \ldots + p_{m-1} R^{\frac{m-1}{m}}, \tag{34}$$

with m a prime number and $p, p_1, \ldots, p_{m-1}, R$ of an analogous form ($R^{\frac{1}{m}}$ is a chosen exterior radical in a hypothetical solution by radicals and it is supposed that it is not a rational function of $a, b, \ldots, p, p_1, \ldots$), Abel first replaces R by $\frac{R}{p_1^m}$ in order to have an expression of the same form with $p_1 = 1$. Putting (34) in the equation, he gets a relation $P = q + q_1 R^{\frac{1}{m}} + q_2 R^{\frac{2}{m}} + \ldots + q_{m-1} R^{\frac{m-1}{m}} = 0$, with coefficients $q, q_1, \ldots$ polynomial in $a, b, c, d, e, p, p_2, \ldots, R$. These coefficients are necessarily 0 for otherwise the two equations $z^m - R = 0$ and $q + q_1 z + \ldots + q_{m-1} z^{m-1} = 0$ would have some common roots, given by the annulation of the greatest common divisor

$$r + r_1 z + \ldots + r_k z^k$$

of their first members. Since the roots of $z^m - R = 0$ are of the form $\alpha_\mu z$, where z is one of them and α_μ is an m-th root of 1, we get a system of k equations $r + \alpha_\mu r_1 z + \ldots + \alpha_\mu^k r_k z^k = 0$ ($0 \le \mu \le k-1$ and $\alpha_0 = 1$), from which it is possible to express z as a rational function of $r, r_1, \ldots$ (and the α_μ). Now the r_k are rational with respect to $a, b, \ldots, R, p, p_2, \ldots$ and we get a contradiction for, by hypothesis, z is not rational with respect to these quantities.

The relation $P = 0$ being identical, the expression (34) is still a root of (33) when $R^{\frac{1}{m}}$ is replaced by $\alpha R^{\frac{1}{m}}$, α an arbitrary m-root of 1, and it is easy to see that the m expressions so obtained are distinct; it results that $m \le 5$. Then (34) gives us m roots y_k ($1 \le k \le m$) of (33), with $R^{\frac{1}{m}}, \alpha R^{\frac{1}{m}}, \ldots, \alpha^{m-1} R^{\frac{1}{m}}$ in place of $R^{\frac{1}{m}}$, and we have

$$\begin{aligned}
p &= \frac{1}{m}(y_1 + y_2 + \ldots + y_m),\\
R^{\frac{1}{m}} &= \frac{1}{m}(y_1 + \alpha^{m-1} y_2 + \ldots + \alpha y_m),\\
p_2 R^{\frac{2}{m}} &= \frac{1}{m}(y_1 + \alpha^{m-2} y_2 + \ldots + \alpha^2 y_m),\\
&\ldots,\\
p_{m-1} R^{\frac{m-1}{m}} &= \frac{1}{m}(y_1 + \alpha y_2 + \ldots + \alpha^{m-1} y_m);
\end{aligned}$$

this proves that $p, p_2, \ldots, p_{m-1}$ and $R^{\frac{1}{m}}$ are rational functions of the roots of (33) (and α). Now if, for instance $R = S + v^{\frac{1}{n}} + S_2 v^{\frac{2}{n}} + \ldots + S_{n-1} v^{\frac{n-1}{n}}$, the same reasoning shows that $v^{\frac{1}{n}}, S, S_2, \ldots$ are rational functions of the roots of (33) and continuing in this manner, we see that every irrational quantity in (34) is a rational function of the roots of (33) (and some roots of 1).

Abel next shows that the innermost radicals in (34) must be of index 2. Indeed if $R^{\frac{1}{m}} = r$ is such a radical, r is a rational function of the 5 roots $y_1, y_2, \ldots, y_5$ and R is a *symmetric* rational function of the same roots, which may be considered as independent variables for (33) is the general quintic equation. So we may arbitrarily permute the y_k in the relation $R^{\frac{1}{m}} = r$ and we see that r takes m different values; a result of Cauchy (1815) now says that $m = 5$ or 2 and the value 5 is easily excluded. We thus know that r takes 2 values and, following Cauchy, it has the form $v(y_1 - y_2)(y_1 - y_3) \ldots (y_2 - y_3) \ldots (y_4 - y_5) = vS^{\frac{1}{2}}$, where v is symmetric and S is the discriminant of (33).

The next radicals are of the form $r = \left(p + p_1 S^{\frac{1}{2}}\right)^{\frac{1}{m}}$, with p, p_1 symmetric. If $r_1 = \left(p - p_1 S^{\frac{1}{2}}\right)^{\frac{1}{m}}$ is the conjugate of r, then $rr_1 = \left(p^2 - p_1^2 S^{\frac{1}{2}}\right)^{\frac{1}{m}} = v$ must be symmetric (otherwise m would be equal to 2 and r would take on 4 values, which is not possible). Thus

$$r + r_1 = \left(p + p_1 S^{\frac{1}{2}}\right)^{\frac{1}{m}} + v\left(p + p_1 S^{\frac{1}{2}}\right)^{-\frac{1}{m}} = z$$

takes m values which implies that $m = 5$ and $z = q + q_1 y + q_2 y^2 + q_3 y^3 + q_4 y^4$, with $q, q_1, \ldots$ symmetric. Combining this relation with (33), we get y rationally in z, a, b, c, d and e, and so of the form

$$y = P + R^{\frac{1}{5}} + P_2 R^{\frac{2}{5}} + P_3 R^{\frac{3}{5}} + P_4 R^{\frac{4}{5}}, \tag{35}$$

with P, R, P_2, P_3 and P_4 of the form $p + p_1 S^{\frac{1}{2}}$, p, p_1 and S rational in a, b, c, d and e. From (35) Abel draws $R^{\frac{1}{5}} = \frac{1}{5}(y_1 + \alpha^4 y_2 + \alpha^3 y_3 + \alpha^2 y_4 + \alpha y_5) = \left(p + p_1 S^{\frac{1}{2}}\right)^{\frac{1}{5}}$, where α is an imaginary fifth root of 1; this is impossible for the first expression takes 120 values and the second only 10.

Euler (1764) had conjectured a form analogous to (35) for the solutions of the quintic equation, with R given by an equation of degree 4. In a letter to Holmboe (24 October 1826, *Œuvres*, t. II, p. 260), Abel states that if a quintic equation is algebraically solvable, its solution has the form $x = A + \sqrt[5]{R} + \sqrt[5]{R'} + \sqrt[5]{R''} + \sqrt[5]{R'''}$ where R, R', R'', R''' are roots of an equation of degree 4 solvable by quadratic radicals; this is explained in a letter to Crelle (14 March 1826, *Œuvres*, t. II, p. 266) for the case of a solvable quintic equation with *rational* coefficients, the solution being $x = c + Aa^{\frac{1}{5}} a_1^{\frac{2}{5}} a_2^{\frac{4}{5}} a_3^{\frac{3}{5}} + A_1 a_1^{\frac{1}{5}} a_2^{\frac{2}{5}} a_3^{\frac{4}{5}} a^{\frac{3}{5}} + A_2 a_2^{\frac{1}{5}} a_3^{\frac{2}{5}} a^{\frac{4}{5}} a_1^{\frac{3}{5}} + A_3 a_3^{\frac{1}{5}} a^{\frac{2}{5}} a_1^{\frac{4}{5}} a_2^{\frac{3}{5}}$, where

$$a = m + n\sqrt{1+e^2} + \sqrt{h(1+e^2+\sqrt{1+e^2})},$$
$$a_1 = m - n\sqrt{1+e^2} + \sqrt{h(1+e^2-\sqrt{1+e^2})},$$
$$a_2 = m + n\sqrt{1+e^2} - \sqrt{h(1+e^2+\sqrt{1+e^2})},$$
$$a_3 = m - n\sqrt{1+e^2} - \sqrt{h(1+e^2-\sqrt{1+e^2})},$$

$A = K + K'a + K''a_2 + K'''aa_2$, $A_1 = K + K'a_1 + K''a_3 + K'''a_1a_3$, $A_2 = K + K'a_2 + K''a + K'''aa_2$ and $A_3 = K + K'a_3 + K''a_1 + K'''a_1a_3$, and $c, h, e, m, n, K, K', K''$ and K''' are rational numbers.

Abel published a new version of his theorem in the first volume of *Crelle's Journal* (1826, *Œuvres*, t. I, p. 66–87). In the first paragraph of this paper, Abel defines the *algebraic functions* of a set of variables $x', x'', x''', \ldots$ They are built from these variables and some constant quantities by the operations of addition, multiplication, division and extraction of roots of prime index. Such a function is *integral* when only addition and multiplication are used, and is then a sum of monomials $Ax'^{m_1}x''^{m_2}\ldots$ It is *rational* when division is also used, but not the extraction of roots, and is then a quotient of two integral functions. The general algebraic functions are classified in *orders*, according to the number of superposed radicals in their expression; a function $f(r', r'', \ldots, \sqrt[n']{p'}, \sqrt[n'']{p''}, \ldots)$ of order μ, with $r', r'', \ldots, p', p'', \ldots$ of order $< \mu$ and f rational, such that none of the $\sqrt[n_k]{p_k}$ is a rational function of the r and the other $\sqrt[n_\ell]{p_\ell}$, is said to be of *degree m* if it contains m radicals $\sqrt[n_k]{p_k}$. Such a function may be written $f(r', r'', \ldots, \sqrt[n]{p})$ with p of order $\mu - 1$, $r', r'', \ldots$ of order $\leq \mu$ and degree $\leq m-1$, and f rational; it is then easy to reduce it to the form $q_0 + q_1 p^{\frac{1}{n}} + q_2 p^{\frac{2}{n}} + \ldots + q_{n-1} p^{\frac{n-1}{n}}$, with coefficients $q_0, q_1, q_2, \ldots$ rational functions of $p, r', r'', \ldots$, so of order $\leq \mu$ and degree $\leq m-1$, $p^{\frac{1}{n}}$ not a rational function of these quantities. Abel carries out the supplementary reduction to the case $q_1 = 1$. In order to do this, he chooses an index μ such that $q_\mu \neq 0$ and puts $q_\mu^n p^\mu = p_1$, which will play the role of p. The starting point of his preceding paper has been completely justified.

In the second paragraph, Abel proves that if an equation is algebraically solvable, one may write its solution in a form in which all the constituent algebraic expressions are rational functions of the roots of the equation. The proof is more precise than that of the 1824 paper, but follows the same lines. The coefficients of the equation are supposed to be rational functions of certain independent variables $x', x'', x''', \ldots$

In the third paragraph, Abel reproduces the proof of Lagrange's theorem (1771) according to which the number of values that a rational function v of n letters may take under the $n!$ substitutions of these letters is necessarily a divisor of $n!$ and Cauchy's theorem (1815) which says that if p is the greatest prime number $\leq n$, and if v takes less than p values, then it takes 1 or 2 values. Indeed it must be invariant for any cycle of p letters, and it is possible to deduce from this that it is invariant for any cycle of 3 letters and from this by any even substitution. Thus, as Ruffini had proved, a rational function of 5 variable cannot take 3 or 4 values. Abel then gives, following Cauchy, the form of a function v of 5 letters $x_1, x_2, \ldots, x_5$ which takes 2 values: it may be written $p + q\rho$, where p and q are rational symmetric functions and

$$\rho = (x_1 - x_2)(x_1 - x_3) \dots (x_4 - x_5)$$

is the square root of the discriminant. Indeed if v_1 and v_2 are the two values of v, $v_1 + v_2 = t$ and $(v_1 v_2)\rho = t_1$ are symmetric and $v_1 = \frac{1}{2}t + \frac{t_1}{2\rho^2}\rho$. Finally, Abel gives the form of a function of 5 quantities which takes 5 values: it is $r_0 + r_1 x + r_2 x^2 + r_3 x^3 + r_4 x^4$, where r_0, r_1, r_2, r_3 and r_4 are symmetric functions of the five quantities and x is one of them. Indeed this is true for a rational function of x_1, x_2, x_3, x_4, x_5 which is symmetric with respect to x_2, x_3, x_4, x_5. Now if v is a function which takes 5 values v_1, v_2, v_3, v_4, v_5 under the substitutions of x_1, x_2, x_3, x_4, x_5, the number μ of values of $x_1^m v$ under the substitutions of x_2, x_3, x_4, x_5 is less than 5, otherwise it would give 25 values under the substitutions of x_1, x_2, x_3, x_4, x_5 and 25 does not divide 5!. If $\mu = 1$, v is symmetric with respect to x_2, x_3, x_4, x_5 and the result is true; it is also true if $\mu = 4$ for the sum $v_1 + v_2 + v_3 + v_4 + v_5$ is completely symmetric and $v_1 + v_2 + v_3 + v_4$ is symmetric with respect to x_2, x_3, x_4, x_5, so $v_5 = v_1 + v_2 + v_3 + v_4 + v_5 - (v_1 + v_2 + v_3 + v_4)$ is of the desired form. It is somewhat more work to prove that μ cannot be 2 or 3. Eliminating x between the equations

$$(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5) = x^5 - ax^4 + bx^3 - cx^2 + dx - e = 0$$

and $r_0 + r_1 x + r_2 x^2 + r_3 x^3 + r_4 x^4 = v$ (a quantity taking 5 values), one obtains

$$x = s_0 + s_1 v + s_2 v^2 + s_3 v^3 + s_4 v^4$$

where s_0, s_1, s_2, s_3 and s_4 are symmetric functions. The paragraph ends with the following lemma: if a rational function v of the 5 roots takes m values under the substitutions of these roots, it is a root of an equation of degree m with coeffficients rational symmetric and it cannot be a root of such an equation of degree less than m.

The fourth paragraph finally gives the proof of the impossibility of a solution by radicals. As in the preceding paper, Abel proves that an innermost radical $R^{\frac{1}{m}} = v$ in a hypothetical solution has an index m (supposed prime) equal to 2 or 5; if $m = 5$, one may write

$$x = s_0 + s_1 R^{\frac{1}{5}} + s_2 R^{\frac{2}{5}} + s_3 R^{\frac{3}{5}} + s_4 R^{\frac{4}{5}}$$

$$\text{and} \quad s_1 R^{\frac{1}{5}} = \frac{1}{5}(x_1 + \alpha^4 x_2 + \alpha^3 x_3 + \alpha^2 x_4 + \alpha x_5)$$

where α is a fifth root of 1, and the second member takes 120 values, which is impossible for it is a root of the equation $z^5 - s_1^5 R = 0$. So $m = 2$ and $\sqrt{R} = p + qs$ with p, q symmetric and $s = (x_1 - x_2) \cdots (x_4 - x_5)$; the second value is $-\sqrt{R} = p - qs$, so $p = 0$. Then, at the second order appear radicals $\sqrt[5]{\alpha + \beta\sqrt{s^2}} = R^{\frac{1}{5}}$ with α, β symmetric as well as $\gamma = \sqrt[5]{\alpha^2 - \beta^2 s^2}$; $p = \sqrt[5]{R} + \frac{\gamma}{\sqrt[5]{R}}$ takes 5 values so that

$$x = s_0 + s_1 p + s_2 p^2 + s_3 p^3 + s_4 p^4 = t_0 + t_1 R^{\frac{1}{5}} + t_2 R^{\frac{2}{5}} + t_3 R^{\frac{3}{5}} + t_4 R^{\frac{4}{5}}$$

with t_0, t_1, t_2, t_3 and t_4 rational in a, b, c, d, e and R. From this relation, one deduces

$$t_1 R^{\frac{1}{5}} = \frac{1}{5}(x_1 + \alpha^4 x_2 + \alpha^3 x_3 + \alpha^2 x_4 + \alpha x_5) = p',$$

where α is a fifth root of 1; $p'^5 = t_1^5 R = u + u'\sqrt{s^2}$ and $(p'^5 - u)^2 = u'^2 s^2$, an equation of degree 10 in p', whereas p' takes 120 values, a contradiction.

Abel reproduced this demonstration in the *Bulletin de Férussac* (1826, t. 6, *Œuvres*, t. I, p. 8794).

In a short paper published in the *Annales de Gergonne* (1827, t. XVII, *Œuvres*, t. I, p. 212–218), Abel treated a problem of the theory of elimination: given two algebraic equations

$$\varphi y = p_0 + p_1 y + p_2 y^2 + \ldots + p_{m-1} y^{m-1} + y^m = 0$$
$$\text{and} \quad \psi y = q_0 + q_1 y + q_2 y^2 + \ldots + q_{n-1} y^{n-1} + y^n$$

with exactly one common solution y, compute any rational function fy of this solution rationally as a function of $p_0, p_1, \ldots, p_{m-1}, q_0, q_1, \ldots, q_{n-1}$. He denotes the roots of ψ by $y, y_1, \ldots, y_{n-1}$ and the product of the φy_j with $j \neq k$ by R_k ($y_0 = y$). As $\varphi y = 0$, $R_k = 0$ for $k \geqslant 1$ so that $fy = \frac{\sum fy_k \cdot \theta y_k \cdot R_k}{\sum \theta y_k \cdot R_k}$, where θ is any rational function. If $fy = \frac{Fy}{\chi y}$, with F and χ polynomial, one may take $\theta = \chi$ to get $fy = \frac{\sum F y_k \cdot R_k}{\sum \chi y_k \cdot R_k}$.

Abel proposes a better solution, based on the observation that R, being a symmetric function of $y_1, y_2, \ldots, y_{n-1}$, may be expressed as $R = \rho_0 + \rho_1 y + \rho_2 y^2 + \ldots + \rho_{n-1} y^{n-1}$, with coefficients $\rho_0, \rho_1, \rho_2, \ldots, \rho_{n-1}$ polynomial in $p_0, p_1, \ldots, p_{m-1}, q_0, q_1, \ldots, q_{n-1}$, and the same is true for $Fy \cdot R = t_0 + t_1 y + t_2 y^2 + \ldots + t_{n-1} y^{n-1}$. Naturally, $R_k = \rho_0 + \rho_1 y_k + \rho_2 y_k^2 + \ldots + \rho_{n-1} y_k^{n-1}$ and $Fy_k \cdot R_k = t_0 + t_1 y_k + t_2 y_k^2 + \ldots + t_{n-1} y_k^{n-1}$. Now taking $\theta y = \frac{1}{\psi' y}$, we have

$$\begin{aligned}\sum \frac{R_k}{\psi' y_k} &= \rho_0 \sum \frac{1}{\psi' y_k} + \rho_1 \sum \frac{y_k}{\psi' y_k} + \rho_2 \sum \frac{y_k^2}{\psi' y_k} + \ldots + \rho_{n-1} \sum \frac{y_k^{n-1}}{\psi' y_k} \\ &= \rho_{n-1}\end{aligned}$$

and, in the same way, $\sum \frac{R_k \cdot F y_k}{\psi' y_k} = t_{n-1}$, so that $Fy = \frac{t_{n-1}}{\rho_{n-1}}$. For a rational function $\frac{Fy}{F'y}$ (where F' is not the derivative of F!), the value is $\frac{t_{n-1}}{t'_{n-1}}$, where t'_{n-1} is the coefficient of y^{n-1} in $F'y \cdot R$. In the case of $fy = y$, let $R = \rho y^{n-1} + \rho' y^{n-2} + \ldots$; then

$$Ry = \rho y^n + \rho' y^{n-1} + \ldots = (\rho' - \rho q_{n-1}) y^{n-1} + \ldots$$

so that $y = -q_{n-1} + \frac{\rho'}{\rho}$.

In his researches about elliptic functions, published in the second and the third volume of *Crelle's Journal* (1827–28, *Œuvres*, t. I, p. 294–314 and 355–362) Abel met some new cases of algebraic equations solvable by radicals. Such equations were known from the time of A. de Moivre (1707), who showed that the equation of degree n (odd integer) giving $\sin \frac{a}{n}$ is solvable by radicals, when $\sin a$ is known, the formula involving $\sin \frac{2\pi}{n}$ in its coefficients. Then Gauss (1801) proved that the cyclotomic equation of degree $n-1$ (n a prime number) giving the n-th root $e^{\frac{2\pi i}{n}}$ is also solvable.

Abel developed an analogous theory for elliptic functions instead of circular functions. Let $x = \varphi u$ be defined by $u = \int_0^x \frac{dt}{\sqrt{(1-c^2t^2)(1+e^2t^2)}}$ (e, c given real parameters); then φ is a uniform meromorphic function in the complex domain, with two independent periods

$$2\omega = 4\int_0^{1/c} \frac{dx}{\sqrt{(1-c^2x^2)(1+e^2x^2)}},$$

$$2i\varpi = 4i\int_0^{1/e} \frac{dx}{\sqrt{(1-e^2x^2)(1+c^2x^2)}}.$$

Abel discovered that the equation of degree n^2 giving $\varphi\left(\frac{a}{n}\right)$ when $\varphi(a)$ is known is solvable by radicals, the formula involving $\varphi\left(\frac{\Omega}{n}\right)$, Ω being a period, in its coefficients. The equation of degree $\frac{n^2-1}{2}$ giving the non-zero values of $\varphi^2\left(\frac{\Omega}{n}\right)$ may be decomposed in $n+1$ equations of degree $\frac{n-1}{2}$, all solvable, by means of an equation of degree $n+1$, which, in general, is not solvable by radicals. For certain singular *moduli* $\frac{e}{c}$, for instance when $\frac{e}{c} = 1, \sqrt{3}$ or $2 \pm \sqrt{3}$, the equation of degree $n+1$ is also solvable; Gauss already knew this lemniscatic case, where $c = e = 1$.

The base for these results is Euler's theorem of addition for elliptic integrals, which gives, in Abel's notation $\varphi(\alpha+\beta) = \frac{\varphi\alpha\, f\beta F\beta + \varphi\beta\, f\alpha F\alpha}{1+e^2c^2\varphi^2\alpha\varphi^2\beta}$, where $f\alpha = \sqrt{1-c^2\varphi^2\alpha}$, $F\alpha = \sqrt{1+e^2\varphi^2\alpha}$. So the roots of the equation for $\varphi\left(\frac{\alpha}{n}\right)$ are $\varphi\left((-1)^{m+\mu}\frac{\alpha}{n} + \frac{m\omega+\mu\varpi i}{n}\right)$, $|m|, |\mu| \leq \frac{n-1}{2}$ and they are rational functions of $\varphi(\beta)$, $f\beta$, $F\beta$, where $\beta = \frac{\alpha}{n}$. Abel defines

$$\varphi_1\beta = \sum_{|m|\leq\frac{n-1}{2}} \varphi\left(\beta + \frac{2m\omega}{n}\right), \quad \psi\beta = \sum_{|\mu|\leq\frac{n-1}{2}} \theta^{\mu}\varphi_1\left(\beta + \frac{2\mu\varpi i}{n}\right)$$

$$\text{and } \psi_1\beta = \sum_{|\mu|\leq\frac{n-1}{2}} \theta^{\mu}\varphi_1\left(\beta - \frac{2\mu\varpi i}{n}\right),$$

where θ is an n-th root of 1; he proves, by means of the addition theorem, that $\psi\beta\cdot\psi_1\beta$ and $(\psi\beta)^n + (\psi_1\beta)^n$ are rational functions of $\varphi\alpha$, so that $\psi\beta = \sqrt[n]{A+\sqrt{A^2-B^n}}$ with A and B rational in $\varphi\alpha$. Indeed, $\varphi_1\beta = \varphi\beta + \sum_{m=1}^{\frac{n-1}{2}} \left(\varphi\left(\beta+\frac{2m\omega}{n}\right) + \varphi\left(\beta - \frac{2m\omega}{n}\right)\right)$ is rational with respect to $\varphi\beta = x$ and $\varphi_1\left(\beta \pm \frac{2\mu\varpi i}{n}\right) = R_\mu \pm R'_\mu\sqrt{(1-c^2x^2)(1+e^2x^2)}$, where R_μ and R'_μ are rational in x, so that $\psi\beta$ and $\psi_1\beta$ have the same form and $\psi_1\beta$ is deduced from $\psi\beta$ by changing the sign of the radical $\sqrt{(1-c^2x^2)(1+e^2x^2)}$. Now $\varphi_1\left(\beta + \frac{2k\omega}{n}\right) = \varphi_1\beta$, $\psi\left(\beta + \frac{2k'\varpi i}{n} + \frac{2k\omega}{n}\right) = \theta^{-k'}\psi\beta$ and $\psi_1\left(\beta + \frac{2k'\varpi i}{n} + \frac{2k\omega}{n}\right) = \theta^{k'}\psi_1\beta$ so that $\psi\beta\cdot\psi_1\beta$ and $(\psi\beta)^n + (\psi_1\beta)^n$, which are rational in $\varphi\beta$, take the same value when $\varphi\beta$ is replaced by any other root of the considered equation and are therefore rational in $\varphi\alpha$. The $n-1$ different values of θ give $n-1$ values A_j and B_j for A and B and one has

$$\varphi_1\left(\beta + \frac{2k\varpi i}{n}\right) = \varphi\alpha + \frac{1}{n}\sum_j \theta_j^{-k}\sqrt[n]{A_j + \sqrt{A_j^2 - B_j^n}}. \tag{36}$$

Then Abel uses $\psi_2\beta = \sum_{|m|\le\frac{n-1}{2}} \theta^m\varphi_1\left(\beta + \frac{2m\omega}{n}\right)$, $\psi_3\beta = \sum_{|m|\le\frac{n-1}{2}} \theta^m\varphi_1\left(\beta - \frac{2m\omega}{n}\right)$, such that $\psi_2\beta\cdot\psi_3\beta$ and $(\psi_2\beta)^n + (\psi_3\beta)^n$ are rational functions of $\varphi_1\beta$. He gets $\psi_2\beta = \sqrt[n]{C + \sqrt{C^2 - D^n}}$ with C, D rational in $\varphi_1\beta$ and

$$\varphi\beta = \frac{1}{n}\left(\varphi_1\beta + \sum_j \sqrt[n]{C_j + \sqrt{C_j^2 - D_j^n}}\right). \tag{37}$$

The radicals in (36) and (37) are not independent (otherwise each formula should give n^{n-1} different values). Indeed, if $\psi^k\beta = \sum_{|\mu|\le\frac{n-1}{2}} \theta^{k\mu}\varphi_1\left(\beta + \frac{2\mu\varpi i}{n}\right)$, $\psi_1^k\beta = \sum_{|\mu|\le\frac{n-1}{2}} \theta^{k\mu}\varphi_1\left(\beta - \frac{2\mu\varpi i}{n}\right)$, where $\theta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, $\psi^k\left(\beta + \frac{2\nu\varpi i}{n}\right) = \theta^{-k\nu}\psi^k\beta$ and $\psi_1^k\left(\beta + \frac{2\nu\varpi i}{n}\right) = \theta^{k\nu}\psi_1^k\beta$ so that $\frac{\psi^k\beta}{(\psi^1\beta)^k} + \frac{\psi_1^k\beta}{(\psi_1^1\beta)^k}$ and $\frac{\psi^k\beta}{(\psi^1\beta)^{k-n}} + \frac{\psi_1^k\beta}{(\psi_1^1\beta)^{k-n}}$ are rational functions of $\varphi\alpha$. As

$$(\psi^1\beta)^n = A_1 + \sqrt{A_1^2 - B_1^n} \quad \text{and} \quad (\psi_1^1\beta)^n = A_1 - \sqrt{A_1^2 - B_1^n},$$

it is easy to deduce $\psi^k\beta = (\psi^1\beta)^k\cdot(F_k + H_k\sqrt{A_1^2 - B_1^n})$, with F_k and H_k rational in $\varphi\alpha$, that is $\sqrt[n]{A_k + \sqrt{A_k^2 - B_k^n}} = (A_1 + \sqrt{A_1^2 - B_1^n})^{\frac{k}{n}}(F_k + H_k\sqrt{A_1^2 - B_1^n})$. In the same way, $\sqrt[n]{C_k + \sqrt{C_k^2 - D_k^n}} = (C_1 + \sqrt{C_1^2 - D_1^n})^{\frac{k}{n}}(K_k + L_k\sqrt{A_1^2 - B_1^n})$, with K_k and L_k rational in $\varphi\alpha$.

For the other problem (division of the periods by an odd prime n), the roots are

$$\varphi^2\left(\frac{m\omega \pm \mu\varpi i}{n}\right), 0 \le m, \mu \le \frac{n-1}{2}, (m, \mu) \ne (0, 0);$$

Abel groups them according to the points $m{:}\mu$ of the projective line $\boldsymbol{P}_1(\boldsymbol{F}_n)$, which gives $n+1$ groups of $\frac{n-1}{2}$ roots each: $\nu(1, 0)$, $\nu(m, 1)$, $1 \le \nu \le \frac{n-1}{2}$, $0 \le m \le n-1$.

He puts $r_\nu = \varphi^2\left(\frac{\nu\omega}{n}\right), r_{\nu,m} = \varphi^2\left(\nu\frac{m\omega+\varpi i}{n}\right)$. If $p = \psi\left(\varphi^2\left(\frac{\omega'}{n}\right)\right)$ is any rational symmetric function of the $\varphi^2\left(\frac{k\omega'}{n}\right)$, $1 \le k \le \frac{n-1}{2}$, $\omega' = m\omega + \mu\varpi i$, then $\psi r_\nu = \psi r_1$ and $\psi r_{\nu,m} = \psi r_{1,m}$. Consider now the equation of degre $n+1$ of which the roots are ψr_1 and $\psi r_{1,m}$ $(0 \le m \le n-1)$. The sum of the k-th powers of these roots is $\frac{2}{n-1}$ times the sum of the $(\psi r_\nu)^k$ and of the $(\psi r_{\nu,m})^k$, so it is rational with respect to e and c, and the same is true for the coefficients of the considered equation of degree $n+1$. We thus see that the equations of degree $\frac{n-1}{2}$ respectively giving the r_ν and the $r_{\nu,m}$ for m fixed have coefficients given by an equation of degree $n+1$. In fact one such auxiliary equation of degree $n+1$ is sufficient, for when a function such as $p = \psi r_1$ is given, any other such function $q = \theta r_1$ is a rational function of p. Abel proves this by a method due to Lagrange, determining θr_1 and the $\theta r_{1,m}$ as solutions of the linear system

$$(\psi r_1)^k \theta r_1 + (\psi r_{1,0})^k \theta r_{1,0} + (\psi r_{1,1})^k \theta r_{1,1} + \ldots + (\psi r_{1,n-1})^k \theta r_{1,n-1} = s_k$$

where the s_k are easily seen to be symmetric with respect to $r_1, r_2, \ldots, r_{1,0}, \ldots$ and so rational functions of e and c.

In order to solve the $n+1$ equations of degree $\frac{n-1}{2}$, with roots $\varphi^2\left(\frac{k\omega'}{n}\right)$, Abel procedes as Gauss did for the cyclotomy: putting $\varepsilon = \frac{\omega'}{n}$ and α a primitive root *modulo* n, the roots may then be written $\varphi^2(\alpha^\ell \varepsilon)$, $0 \le \ell \le \frac{n-3}{2}$ and the Lagrange resolvant is

$$\psi(\varepsilon) = \varphi^2(\varepsilon) + \varphi^2(\alpha\varepsilon)\theta + \ldots + \varphi^2\left(\alpha^{\frac{n-3}{2}}\varepsilon\right)\theta^{\frac{n-3}{2}},$$

where θ is a $\frac{n-1}{2}$-th root of 1. It is a rational function of $\varphi^2(\varepsilon)$ and its $\frac{n-1}{2}$-th power v is symmetric with respect to the $\frac{n-1}{2}$ roots. Thus v is known when the equation of degree $n+1$ is solved. Varying θ, we get $\frac{n-1}{2}$ values v_k for v and

$$\varphi^2(\varepsilon) = \frac{2}{n-1}\left(-p_{\frac{n-3}{2}} + \sqrt[\frac{n-1}{2}]{v_1} + \ldots + \sqrt[\frac{n-1}{2}]{v_{\frac{n-3}{2}}}\right),$$

where $-p_{\frac{n-3}{2}} = \varphi^2(\varepsilon) + \varphi^2(\alpha\varepsilon) + \ldots + \varphi^2\left(\alpha^{\frac{n-3}{2}}\varepsilon\right)$ corresponds to $\theta = 1$ and is symmetric. The $n-1$ radicals are not independent: $s_k = \frac{\sqrt[\frac{n-1}{2}]{v_k}}{\left(\sqrt[\frac{n-1}{2}]{v_1}\right)^k}$ is a rational function of $\varphi^2(\varepsilon)$ which remains invariant when ε is replaced by $\alpha^m\varepsilon$, so it is a rational function of the root of the auxiliary equation of degree $n+1$.

When $e = c = 1, \omega = \varpi$ and there exists a *complex multiplication*, that is a formula

$$\varphi(m+\mu i)\delta = \varphi\delta \cdot T,$$

for m, μ integers and $m+\mu$ odd, with T a rational function if $\varphi^4(\delta)$. This permits the solution by radicals of the equation giving $\varphi\left(\frac{\omega}{n}\right)$ whenever n is a prime number of the form $4\nu+1$. Indeed, consider $n=\alpha^2+\beta^2$, where α and β are relatively prime integers such that $\alpha+\beta$ is odd, and there exists integers m, t such that $1=2m\alpha-nt$; then $\frac{m\omega}{\alpha+\beta i}+\frac{m\omega}{\alpha-\beta i}=\frac{2m\alpha}{n}$ and

$$\varphi\left(\frac{2m\alpha}{n}\right)=\varphi\left(\frac{\omega}{n}+t\omega\right)=(-1)^t\varphi\left(\frac{\omega}{n}\right),$$

so it is sufficient to consider the equation for $\varphi\left(\frac{\omega}{\alpha+\beta i}\right)=\varphi\delta$. This equation is equivalent to $\varphi(\alpha+\beta i)\delta=0$ and its roots are $x=\varphi\left(\frac{m+\mu i}{\alpha+\beta i}\omega\right)$, m, μ integers; it is easy to reduce these roots to the form $x=\varphi\left(\frac{\rho\omega}{\alpha+\beta i}\right)$, $|\rho|\le\frac{n-1}{2}=2\nu$ and to see that, in this form, they are all distinct. Moreover, the equation has no multiple roots, and finally, since φ is an odd function, we have to consider an equation of degree 2ν, with roots $\varphi^2(\delta), \varphi^2(2\delta), \dots, \varphi^2(2\nu\delta)$ or $\varphi^2(\delta), \varphi^2(\varepsilon\delta), \varphi^2(\varepsilon^2\delta), \dots, \varphi^2(\varepsilon^{2\nu-1}\delta)$, where ε is a primitve root *modulo* n. Then, as for the preceding problem, one sees that $\varphi^2(\varepsilon^m\delta)=\frac{1}{2\nu}\left(A+\theta^{-m}v^{\frac{1}{2\nu}}+s_2\theta^{-2m}v^{\frac{2}{2\nu}}+s_{2\nu-1}\theta^{-(2\nu-1)m}v^{\frac{2\nu-1}{2\nu}}\right)$, where

$$\begin{aligned}
v&=(\varphi^2(\delta)+\theta\varphi^2(\varepsilon\delta)+\theta^2\varphi^2(\varepsilon^2\delta)+\dots+\theta^{2\nu-1}\varphi^2(\varepsilon^{2\nu-1}\delta))^{2\nu},\\
s_k&=\frac{\varphi^2(\delta)+\theta^k\varphi^2(\varepsilon\delta)+\theta^{2k}\varphi^2(\varepsilon^2\delta)+\dots+\theta^{(2\nu-1)k}\varphi^2(\varepsilon^{2\nu-1}\delta)}{(\varphi^2(\delta)+\theta\varphi^2(\varepsilon\delta)+\theta^2\varphi^2(\varepsilon^2\delta)+\dots+\theta^{2\nu-1}\varphi^2(\varepsilon^{2\nu-1}\delta))^k},\\
A&=\varphi^2(\delta)+\varphi^2(\varepsilon\delta)+\varphi^2(\varepsilon^2\delta)+\dots+\varphi^2(\varepsilon^{2\nu-1}\delta)
\end{aligned}$$

are rational with respect to the coefficients of the equation, so of the form $a+bi$ with a and b rational numbers. Abel oberves that, when n is a Fermat prime number 2^N+1, all the radicals in the solution are of index 2 for $2\nu=2^{N-1}$ and $\theta^{2^{N-1}}=1$. He applies these results to the division of the lemniscate of polar equation $x=\sqrt{\cos 2\theta}$ (x distance to the origin, θ polar angle), for which the elementary arc is $\frac{dx}{\sqrt{1-x^4}}$.

All these examples of solvable equations (Moivre, Gauss, elliptic functions) gave Abel models for a general class of solvable equations; following Kronecker, we call them Abelian equations and they are the object of a memoir published in the fourth volume of *Crelle's Journal* (1829, *Œuvres*, t. I, p. 478–507). To begin with, Abel defines (in a footnote) the notion of an irreducible equation with coefficients rational functions of some quantities $a, b, c, \dots$ considered as known; his first theorem states that if a root of an irreducible equation $\varphi x=0$ annihilates a rational function fx of x and the same quantities $a, b, c, \dots$, then the it is still true for any other root of $\varphi x=0$ (the proof is given in a footnote).

The second theorem states that if an irreducible equation $\varphi x=0$ of degree μ has two roots x' and x_1 related by a rational relation $x'=\theta x_1$ with known coefficients, then the given equation may be decomposed in m equations of degree n of which the coefficients are rational functions of a root of an auxiliary equation of degree m (naturally $\mu=mn$). First of all, the equation $\varphi(\theta x_1)=0$ with the theorem I shows that $\varphi(\theta x)=0$ for any root x of $\varphi x=0$; so $\theta x'=\theta^2 x_1, \theta^3 x_1, \dots$ are all roots of $\varphi x=0$. If $\theta^m x_1=\theta^{m+n}x_1$ (the equation has only a finite number of roots), or $\theta^n(\theta^m x_1) - \theta^m x_1 = 0$, we have $\theta^n x - x = 0$ for any root of $\varphi x = 0$ by the theorem I and, in particular $\theta^n x_1 = x_1$; if n is minimal with this property, $x_1, \theta x, \dots, \theta^{n-1}x_1$ are distinct roots and the sequence $(\theta^m x_1)$ is periodic with period n. When $\mu > n$, there exists a root x_2 which does not belong to this sequence; then $(\theta^m x_2)$ is a new sequence of roots with exactly the same period for $\theta^n x_2 = x_2$ and if $\theta^k x_2 = x_2$ for a $k < n$, we should have $\theta^k x_1 = x_1$. When $\mu > 2n$, there exists a root x_3 different from the $\theta^m x_1$ and the $\theta^m x_2$, and the sequence $(\theta^m x_3)$ has a period n. Continuing in this way, we see that μ is necessarily a multiple mn of n and that the μ roots may be grouped in m sequences $(\theta^k x_j)_{0 \leq k \leq n-1}$ $(j = 1, 2, \dots, m)$. Note that this proof is analogous to that of Lagrange establishing that a rational function of n letters takes, under the substitutions of these letters, a number of values which divides $n!$. In order to prove his second theorem, Abel considers a rational symmetric function $y_1 = f(x_1, \theta x_1, \dots, \theta^{n-1}x_1) = Fx_1$ of the first n roots and the corresponding $y_j = Fx_j = F(\theta^k x_j)$ $(2 \leq j \leq m)$; for any natural integer ν, the sum $y_1^\nu + y_2^\nu + \dots + y_m^\nu$ is symmetric with respect to the mn roots of $\varphi x = 0$, so it is a known quantity and the same is true for the coefficients of the equation with the roots $y_1, y_2, \dots, y_m$. Since the equation with the roots $x_1, \theta x_1, \dots, \theta^{n-1}x_1$ has its coefficients rational symmetric functions of $x_1, \theta x_1, \dots, \theta^{n-1}x_1$, each of these coefficients is a root of an equation of degree m with known coefficients. In fact one auxiliary equation of degree m is sufficient: this is proved by the stratagem of Lagrange already used by Abel for the division of the periods of elliptic functions (Abel notes that it is necessary to choose the auxiliary equation without multiple roots, which is always possible).

When $m = 1$, $\mu = n$; the roots constitute only one sequence $x_1, \theta x_1, \dots, \theta^{\mu-1}x_1$ and the equation $\varphi x = 0$ is algebraically solvable as Abel states it in his theorem III. We now say that the equation $\varphi x = 0$ is *cyclic*. This result comes from the fact that the Lagrange resolvant

$$x + \alpha\theta x + \alpha^2\theta^2 x + \dots + \alpha^{\mu-1}\theta^{\mu-1}x$$

(x any root of the equation, α μ-th root of 1) has a μ-th power v symmetric with respect to the μ roots. We now get $x = \frac{1}{\mu}(-A + \sqrt[\mu]{v_1} + \sqrt[\mu]{v_2} + \dots + \sqrt[\mu]{v_{\mu-1}})$, where $v_0, v_1, \dots, v_{\mu-1}$ are the values of v corresponding to the diverse μ-th roots α of 1 ($\alpha = 1$ for v_0) and $-A = \sqrt[\mu]{v_0}$. The $\mu - 1$ radicals are not independent for if $\alpha = \cos\frac{2\pi}{\mu} + \sqrt{-1}\sin\frac{2\pi}{\mu}$ and if

$$\sqrt[\mu]{v_k} = x + \alpha^k\theta x + \alpha^{2k}\theta^2 x + \dots + \alpha^{(m-1)k}\theta^{\mu-1}x,$$

the quantity $\sqrt[\mu]{v_k}(\sqrt[\mu]{v_1})^{\mu-k} = a_k$ is a symmetric function of the roots of $\varphi x = 0$, so it is known. Abel recalls that this method was used by Gauss in order to solve the equations of the cyclotomy. The theorem IV is a corollary of the preceding one: when the degree μ is a prime number and two roots of $\varphi x = 0$ are such that one of them is a rational function of the other, then the equation is algebraically solvable.

As $a_{\mu-1} = \sqrt[\mu]{v_{\mu-1}} \cdot \sqrt[\mu]{v_1} = a$, does not change when α is replaced by its complex conjugate, it is real when the known quantities are supposed to be real. Thus v_1 and $v_{\mu-1}$ are complex conjugate and

$$v_1 = c + \sqrt{-1}\sqrt{a^\mu - c^2} = (\sqrt{\rho})^\mu(\cos\delta + \sqrt{-1}\cdot\sin\delta),$$

and so

$$\sqrt[\mu]{v_1} = \sqrt{\rho}\cdot\left(\cos\frac{\delta + 2m\pi}{\mu} + \sqrt{-1}\cdot\sin\frac{\delta + 2m\pi}{\mu}\right).$$

So in order to solve $\varphi x = 0$ it suffices to divide the circle in μ equal parts, to divide the angle δ (which is constructible) in μ equal parts and to extract the square root of ρ. Moreover, Abel notes that the roots of $\varphi x = 0$ are all real or all imaginary; if μ is odd they are all real.

The theorem VI is relative to a cyclic equation $\varphi x = 0$ of composite degree $\mu = m_1 \cdot m_2 \cdots m_\omega = m_1 \cdot p_1$. Abel groups the roots in m_1 sequences $(\theta^{km_1+j}x)_{0\le k\le p_1-1}$ $(0 \le j \le m_1 - 1)$ of p_1 roots each. This allows the decomposition of the equation in m_1 equations of degree p_1 with coefficients rational functions of a root of an auxiliary equation of degree m_1. In the same way, each equation of degree $p_1 = m_2 \cdot p_2$ is decomposed in m_2 equations of degree p_2 using an auxiliary equation of degree m_2, etc. Finally, the solution of $\varphi x = 0$ is reduced to that of ω equations of respective degrees $m_1, m_2, \ldots, m_\omega$. As Abel notes, this is precisely what Gauss did for the cyclotomy. The case in which $m_1, m_2, \ldots, m_\omega$ are relatively prime by pairs is particularly interesting. Here for $1 \le k \le \omega$ an auxiliary equation $fy_k = 0$ of degree m_k allows to decompose $\varphi x = 0$ in m_k equations $F_k(\theta^j x, y_k) = 0$ of degree $n_k = \frac{\mu}{m_k}$ $(0 \le j \le m_k - 1)$. Since x is the only common root of the ω equations $F_k(x, y_k) = 0$ (for $\theta^{km_p}x = \theta^{\ell m_q}x$ with $k \le n_p - 1$ and $\ell \le n_q - 1$ implies $km_p = \ell m_q$ and then $k = \ell = 0$ if $p \ne q$), it is rational with respect to $y_1, y_2, \ldots, y_\omega$. So, in this case, the resolution is reduced to that of the equations $f_1 y_1 = 0, f_2 y_2 = 0, \ldots, f_\omega y_\omega = 0$ of respective degrees $m_1, m_2, \ldots, m_\omega$ and with coefficients known quantities. One may take for the m_k the prime-powers which compose μ.

All the auxiliary equations are cyclic as is $\varphi x = 0$, so they may be solved by the same method. This follows from the fact that if

$$y = Fx = f(x, \theta^m x, \theta^{2m}x, \ldots, \theta^{(n-1)m}x)$$

is symmetric with respect to $x, \theta^m x, \theta^{2m}x, \ldots, \theta^{(n-1)m}x$, so is $F(\theta x)$. Then, by Lagrange's stratagem $F(\theta x)$ is a rational function λy of y.

Abel ends this part of the memoir with the theorem VII, relating to a cyclic equation of degree 2^ω: its solution amounts to the extraction of ω square roots. This is the case for Gauss' division of the circle by a Fermat prime.

The second part deals with algebraic equations of which all the roots are rational functions of one of them, say x. According to Abel's theorem VIII, if $\varphi x = 0$ is such an equation of degree μ and if, for any two roots θx and $\theta_1 x$ the relation $\theta\theta_1 x = \theta_1\theta x = 0$ is true, then the equation is algebraically solvable. Abel begins by observing that one may suppose that $\varphi x = 0$ is irreducible. So that if n is the period of $(\theta^k x)$, the roots are grouped in $m = \frac{\mu}{n}$ groups of n roots. Each group contains the roots of an equation of degree n with coefficients rational functions of a quantity $y = f(x, \theta x, \theta^2 x, \ldots, \theta^{n-1}x)$ given by an equation of degree m

with known coefficients, which is easily seen to be irreducible. The other roots of the equation in y are of the form $y_1 = f(\theta_1 x, \theta\theta_1 x, \theta^2\theta_1 x, \ldots, \theta^{n-1}\theta_1 x) = f(\theta_1 x, \theta_1\theta x, \theta_1\theta^2 x, \ldots, \theta_1\theta^{n-1}x)$ (by the hypothesis), so rational symmetric with respect to $x, \theta x, \theta^2 x, \ldots, \theta^{n-1}x$ and (again by Lagrange's stratagem) rational in y: $y_1 = \lambda y$. Now if $y_2 = \lambda_1 y = f(\theta_2 x, \theta\theta_2 x, \theta^2\theta_2 x, \ldots, \theta^{n-1}\theta_2 x)$,

$$\begin{aligned}\lambda\lambda_1 y = \lambda y_2 &= f(\theta_1\theta_2 x, \theta\theta_1\theta_2 x, \ldots, \theta^{n-1}\theta_1\theta_2 x)\\ &= f(\theta_2\theta_1 x, \theta\theta_2\theta_1 x, \ldots, \theta^{n-1}\theta_2\theta_1 x) = \lambda_1\lambda y\end{aligned}$$

so that the equation in y has the same property as the initial equation $\varphi x = 0$ and it is possible to deal with it in the same manner. Finally $\varphi x = 0$ is solvable through a certain number of cyclic equations of degrees $n, n_1, n_2, \ldots, n_\omega$ such that $\mu = nn_1n_2\cdots n_\omega$, this is Abel's theorem IX. In the theorem X, Abel states that when $\mu = \varepsilon_1^{\nu_1}\varepsilon_2^{\nu_2}\ldots\varepsilon_\alpha^{\nu_\alpha}$ with $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\alpha$ prime, the solution amounts to that of ν_1 equations of degree ε_1, ν_2 equations of degree $\varepsilon_2, \ldots, \nu_\alpha$ equations of degree ε_α, all solvable by radicals.

As an example, Abel applies his general theorem to the division of the circle in $\mu = 2n + 1$ equal parts, where μ is a prime number; the equation with roots $\cos\frac{2\pi}{\mu}, \cos\frac{4\pi}{\mu}, \ldots, \cos\frac{2n\pi}{\mu}$ has rational coefficients and it is cyclic. If m is a primitive root *modulo* μ, the roots are $x, \theta x, \ldots, \theta^{n-1}x$ where $x = \cos\frac{2\pi}{\mu} = \cos a$ and $\theta x = \cos ma$, polynomial of degree m. As Gauss has proved, the division of the circle is reduced in μ parts is reduced to the division of the circle in n parts, the division of a certain (constructible) angle in n parts and the extraction of a square root of a quantity

$$\begin{aligned}\rho = |&(x + \alpha\theta x + \alpha^2\theta^2 x + \ldots + \alpha^{n-1}\theta^{n-1}x)\\ &\times (x + \alpha^{n-1}\theta x + \alpha^{n-2}\theta^2 x + \ldots + \alpha\theta^{n-1}x)|.\end{aligned}$$

It is not difficult to compute $\pm\rho = \frac{1}{2}n - \frac{1}{4} - \frac{1}{2}(\alpha + \alpha^2 + \ldots + \alpha^{n-1}) = \frac{1}{2}n + \frac{1}{4}$ so the square root is $\sqrt{\mu}$ conformally to Gauss' result. After his notebooks, we know that Abel also wanted to apply his theory to the division of periods of elliptic functions with a singular *modulus*, precisely in the case where $\omega = \varpi\sqrt{2n+1}$.

On the 8 of October 1828, Abel sent the statement of three theorems on algebraic equations to Crelle.

A. Given a prime number n and n unknown quantities $x_1, x_2, \ldots, x_n$ related by the relations $\varphi(x_1, x_2, \ldots, x_n) = \varphi(x_2, x_3, \ldots, x_1) = \ldots = \varphi(x_n, x_1, \ldots, x_{n-1}) = 0$, where φ is a polynomial of degree m, the equation of degree $m^n - m$ obtained by elimination of $n - 1$ of the quantities and division by the factor $\varphi(x, x, \ldots, x)$ is decomposable in $\frac{m^n - m}{n}$ equations of degree n, all algebraically solvable, with the help of an equation of degree $\frac{m^n - m}{n}$. Abel gives, as examples, the cases where $n = 2, m = 3$ and $n = 3, m = 2$; in these cases $m^n - m = 6$ and the equation of degree 6 is algebraically solvable.

B. If three roots of an irreducible equation of prime degree are so related that one of them is rationally expressed by the other two, then the equation is algebraically solvable.

This theorem is given, as a necessary and sufficient condition, by E. Galois as an application of his *Mémoire sur les conditions de résolubilité des équations par radicaux* (1831) and it was at first interpreted as the main result of this memoir.

C. If two roots of an irreducible equation of prime degree are so related that one of them is rationally expressed by the other, then the equation is algebraically solvable.

This statement is the same as that of the theorem IV in the 1829 memoir (which was composed in March 1828).

Abel left uncompleted an important paper *Sur la théorie algébrique des équations* (*Œuvres*, t. II, p. 217–243). In the introduction, he explains in a very lucid way his method in mathematics, saying that one must give a problem such a form that it is always possible to solve it. For the case of the solution by radicals of algebraic equations, Abel formulates certain problems:

(1) To find all the equations of a given degree which are algebraically solvable.
(2) To judge whether a given equation is algebraically solvable.
(3) To find all the equations that a given algebraic function may satisfy.

Here an algebraic function is defined, as in the 1826 paper, as built by the operations of addition, subtraction, multiplication, division and extraction of roots of prime index. There are two types of equations to consider: those for which the coefficients are rational functions of certain variables $x, z, z', z'', \dots$ (with arbitrary numerical coefficients; for instance the general equation of a given degree, for which the coefficients are independent variables) and those for which the coefficients are constant; in the last case the coefficients are supposed to be rational expressions in given numerical quantities $\alpha, \beta, \gamma, \dots$ with rational coefficients. An equation of the first type is said to be algebraically satisfied (resp. algebraically solvable) when it is verified when the unknown is replaced by an algebraic function of $x, z, z', z'', \dots$ (resp. when all the roots are algebraic functions of $x, z, z', z'', \dots$); there are analogous definitions for the second type, with "algebraic function of $x, z, z', z'', \dots$" replaced by "algebraic expression of $\alpha, \beta, \gamma, \dots$".

In order to attack his three problems, Abel is led to solve the following ones "To find the most general form of an algebraic expression" and "To find all the possible equations which an algebraic function may satisfy". These equations are infinite in number but, for a given algebraic function, there is one of minimal degree, and this one is irreducible.

Abel states some general results he has obtained about these problems:

(1) If an irreducible equation may be algebraically satisfied, it is algebraically solvable; the same expression represents all the roots, by giving the radicals in it all their values.
(2) If an algebraic expression satisfies an equation, it is possible to give it such a form that it still satisfies the equation when one gives to the radicals in it all their values.
(3) The degree of an irreducible algebraically solvable equation is the product of certain indexes of the radicals in the expression of the roots.

About the problem "To find the most general algebraic expression which may staisfy an equation of given degree", Abel states the following results:

(1) If an irreducible equation of prime degree μ is algebraically solvable, its roots are of the form $y = A + \sqrt[\mu]{R_1} + \sqrt[\mu]{R_2} + \ldots + \sqrt[\mu]{R_{\mu-1}}$, where A is rational and $R_1, R_2, \ldots, R_{\mu-1}$ are roots of an equation of degree $\mu - 1$.
This form was conjectured by Euler (1738) for the general equation of degree μ.
(2) If an irreducible equation of degree μ^α, with μ prime, is algebraically solvable, either it may be decomposed in $\mu^{\alpha-\beta}$ equations of degree μ^β of which the coefficients depend on an equation of degree $\mu^{\alpha-\beta}$, or each root has the form $y = A + \sqrt[\mu]{R_1} + \sqrt[\mu]{R_2} + \ldots + \sqrt[\mu]{R_\nu}$, with A rational and $R_1, R_2, \ldots, R_\nu$ roots of an equation of degree $\nu \leq \mu^\alpha - 1$.
(3) If an irreducible equation of degree μ not a prime-power is algebraically solvable, it is possible to decompose μ in a product of two factors μ_1 and μ_2 and the equation in μ_1 equations of degree μ_2 of which the coefficients depend on equations of degree μ_1.
(4) If an irreducible equation of degree μ^α, with μ prime, is algebraically solvable, its roots may be expressed by the formula $y = f(\sqrt[\mu]{R_1}, \sqrt[\mu]{R_2}, \ldots, \sqrt[\mu]{R_\alpha})$ with f rational symmetric and $R_1, R_2, \ldots, R_\alpha$ roots of an equation of degree $\leq \mu^\alpha - 1$.

A corollary of (3) is that when an irreducible equation of degree $\mu = \mu_1^{\alpha_1}\mu_2^{\alpha_2}\ldots\mu_\omega^{\alpha_\omega}$ ($\mu_1, \mu_2, \ldots, \mu_\omega$ prime) is algebraically solvable, only the radicals necessary to express the roots of equations of degrees $\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \ldots, \mu_\omega^{\alpha_\omega}$ appears in the expression of the roots. Abel adds that if an irreducible equation is algebraically solvable, its roots may be found by Lagrange's method. According to this method, an equation of degree μ is reduced to the solution of $\frac{(\mu-1)!}{\varphi(\mu)}$ equations of degree $\varphi(\mu)$ (φ the Euler function) with the help of an equation of degree $\frac{(\mu-1)!}{\varphi(\mu)}$ (Abel text leaves a blank at the place of these numbers). Abel announces that a necessary condition for the algebraic solvability is that the equation of degree $\frac{(\mu-1)!}{\varphi(\mu)}$ have a root rational with respect to the coefficients of the proposed equation; if μ is a prime number, this condition is also sufficient.

The first paragraph of the paper explains the structure of algebraic expressions, as was done in the published 1826 article; this time, the order of such an expression is defined as the minimum number of radicals necessary to write it. In the second paragraph, a polynomial

$$y^n + Ay^{n-1} + A'y^{n-2} + \ldots = \varphi(y, m)$$

is said to be of order m when the maximum order of its coefficients $A, A', \ldots$ is m. The first theorem states that an expression $t_0 + t_1 y_1^{\frac{1}{\mu_1}} + t_2 y_1^{\frac{2}{\mu_1}} + \ldots + t_{\mu_1-1} y_1^{\frac{\mu_1-1}{\mu_1}}$, with $t_0, t_1, \ldots, t_{\mu_1-1}$ rational with respect to a μ_1-th root ω of 1 and radicals different from $y_1^{\frac{1}{\mu_1}}$, is 0 only if $t_0 = t_1 = \ldots = t_{\mu_1-1} = 0$. The second theorem states that if an equation $\varphi(y, m) = 0$ of order m is satisfied by an algebraic expression

$y = p_0 + p_1 \sqrt[\mu_1]{y_1} + \dots$ of order $n > m$, it is still satisfied by the expression with $\omega \sqrt[\mu_1]{y_1}, \omega^2 \sqrt[\mu_1]{y_1}, \dots$ instead of $\sqrt[\mu_1]{y_1}$, where ω is a μ_1-th root of 1. After the third theorem when two equations $\varphi(y, m) = 0$ and $\varphi_1(y, n) = 0$ have a common root, the first one being irreducible and $n \leq m$, then $\varphi_1(y, n) = f(y, m) \cdot \varphi(y, m)$. Then the fourth theorem says that $\varphi_1(y, n)$ is divisible by the product $\prod \varphi(y, m)$ of $\varphi(y, m)$ and the polynomial $\varphi'(y, m), \varphi''(y, m), \dots, \varphi^{(\mu-1)}(y, m)$ obtained by successively replacing in $\varphi(y, m)$ the outermost radical $\sqrt[\mu]{y_1}$ by $\omega \sqrt[\mu]{y_1}, \omega^2 \sqrt[\mu]{y_1}, \dots$ (ω μ-th root of 1); this comes from the fact that $\varphi'(y, m), \varphi''(y, m), \dots, \varphi^{(\mu-1)}(y, m)$ are relatively prime by pairs. In the fifth theorem, Abel states that if $\varphi(y, m) = 0$ is irreducible, so is $\prod \varphi(y, m) = \varphi_1(y, m') = 0$.

Now if $a_m = f\left(y_m^{\frac{1}{\mu_m}}, y_{m-1}^{\frac{1}{\mu_{m-1}}}, \dots\right)$, of order m, is a root of an irreducible equation $\psi(y) = 0$, ψ must be divisible by $y - a_m$, and so also by $\prod(y - a_m) = \varphi(y, m_1)$ (theorem IV), which is irreducible (theorem V). It now follows that ψ is divisible by $\prod \varphi(y, m_1) = \varphi_1(y, m_2)$ and by $\prod \varphi_1(y, m_2) = \varphi_2(y, m_3)$, etc., with $m > m_1 > m_2 > \dots$ Finally, we arrive at some $m_{\nu+1} = 0$ and $\varphi_\nu(y, 0)$ divides $\psi(y)$ and has rational coefficients, so that $\psi = \varphi_\nu$. This leads to the degree of ψ, for that of $\varphi(y, m_1)$ is μ_m, that of $\varphi_1(y, m_2)$ is $\mu_m \cdot \mu_{m_1}, \dots$ and that of φ_ν is $\mu_m \cdot \mu_{m_1} \dots \mu_{m_\nu} = \mu$. This is the third general result of the introduction, with the further explanation that the index of the outermost radical is always one of the factors of the degree μ. The first general result of the introduction is also a consequence of that fact, as the fact that an algebraic expression solution of an irreducible equation of degree μ takes exactly μ values.

In the third paragraph, Abel first deals with the case in which μ is a prime number; then $\mu_m = \mu$ and $a_m = p_0 + p_1 s^{\frac{1}{\mu}} + p_2 s^{\frac{2}{\mu}} + \dots + p_{\mu-1} s^{\frac{\mu-1}{\mu}}$, with $s = y_m$; giving to the radical $s^{\frac{1}{\mu}}$ its μ values $s^{\frac{1}{\mu}}, \omega s^{\frac{1}{\mu}}, \dots, \omega^{\mu-1} s^{\frac{1}{\mu}}$, where ω is a μ-th root of 1, we get μ values $z_1, z_2, \dots, z_\mu$ for a_m and, as the given equation has only μ roots, we cannot get new values by replacing the p_j or s by other values p'_j and s' obtained by changing the value of the radicals they contain. Now if

$$p'_0 + p'_1 \omega' s'^{\frac{1}{\mu}} + \dots + p'_{\mu-1} \omega'^{\mu-1} s'^{\frac{\mu-1}{\mu}} = p_0 + p_1 \omega s^{\frac{1}{\mu}} + \dots + p_{\mu-1} \omega^{\mu-1} s^{\frac{\mu-1}{\mu}},$$

we see that different values $\omega_0, \omega_1, \dots, \omega_{\mu-1}$ of the root ω correspond to different values of the root ω' of 1. Writing the corresponding equalities and adding, we obtain $\mu p'_0 = \mu p_0$, so $p'_0 = p_0$ and then

$$\mu p'_1 s'^{\frac{1}{\mu}} = p_1 s^{\frac{1}{\mu}} (\omega_0 + \omega_1 \omega^{-1} + \omega_2 \omega^{-2} + \dots + \omega_{\mu-1} \omega^{-\mu+1}) + \dots$$

So

$$s'^{\frac{1}{\mu}} = f\left(\omega, p_0, p'_0, p_1, p'_1, \dots, s', s^{\frac{1}{\mu}}\right) = q_0 + q_1 s^{\frac{1}{\mu}} + q_2 s^{\frac{2}{\mu}} + \dots + q_{\mu-1} s^{\frac{\mu-1}{\mu}}$$

and $s' = t_0 + t_1 s^{\frac{1}{\mu}} + t_2 s^{\frac{2}{\mu}} + \dots + t_{\mu-1} s^{\frac{\mu-1}{\mu}}$, although Abel's given proof of the fact that $t_1 = t_2 = \dots = t_{\mu-1} = 0$ is not quite complete. In the notes at the end of the second volume of Abel's *Works*, Sylow has explained how to complete the proof (p. 332–335) in order to finally obtain $p'^{\mu}_1 s' = p^{\mu}_{\nu} s^{\nu}$ for some ν between 2 and $\mu - 1$; this shows that $p^{\mu}_1 s$ is root of an equation of degree $\leq \mu - 1$.

Changing s, it is possible to get $p_1 = 1$ and then, we have as usual $p_0 = \frac{1}{\mu}(z_1 + z_2 + \ldots + z_\mu)$ a known quantity, $s^{\frac{1}{\mu}} = \frac{1}{\mu}(z_1 + \omega^{\mu-1} z_2 + \ldots + \omega z_\mu)$, $p_2 s^{\frac{2}{\mu}} = \frac{1}{\mu}(z_1 + \omega^{\mu-2} z_2 + \ldots + \omega^2 z_\mu), \ldots$; this gives

$$p_2 s = \left(\frac{1}{\mu}\right)^{\mu-1} (z_1 + \omega^{-2} z_2 + \ldots + \omega^{-2(\mu-1)} z_\mu)(z_1 + \omega^{-1} z_2 + \ldots + \omega^{-(\mu-1)} z_\mu)^{\mu-2},$$

$$p_3 s = \left(\frac{1}{\mu}\right)^{\mu-2} (z_1 + \omega^{-3} z_2 + \ldots + \omega^{-3(\mu-1)} z_\mu)(z_1 + \omega^{-1} z_2 + \ldots + \omega^{-(\mu-1)} z_\mu)^{\mu-3}, \text{ etc.}$$

By the usual Lagrangian stratagem, Abel proves that $q_1 = p_m s$ is a rational function of s and the known quantities for $2 \leq m \leq \mu - 1$. The ν distinct values of s are of the form $p^{\mu}_m s^m$ with $2 \leq m \leq \mu - 1$; Abel shows that the irreducible equation of which s is a root is *cyclic* of degree dividing $\mu - 1$, with roots $s, s_1 = \theta s, \ldots, s_{\nu-1} = \theta^{\nu-1} s$, where $\theta s = (fs)^{\mu} s^{m^{\alpha}}$, f rational, $2 \leq m \leq \mu - 1$ and α a divisor of $\mu - 1$. He finally arrives at the following form for the root z_1 of $\psi y = 0$:

$$\begin{aligned} z_1 = p_0 &+ s^{\frac{1}{\mu}} + s_1^{\frac{1}{\mu}} + \ldots + s_{\nu-1}^{\frac{1}{\mu}} + \varphi_1 s \cdot s^{\frac{m}{\mu}} + \varphi_1 s_1 \cdot s_1^{\frac{m}{\mu}} \\ &+ \ldots + \varphi_1 s_{\nu-1} \cdot s_{\nu-1}^{\frac{m}{\mu}} + \varphi_2 s \cdot s^{\frac{m^2}{\mu}} + \varphi_2 s_1 \cdot s_1^{\frac{m^2}{\mu}} + \ldots + \varphi_2 s_{\nu-1} \cdot s_{\nu-1}^{\frac{m^2}{\mu}} \\ &+ \ldots + \varphi_{\alpha-1} s \cdot s^{\frac{m^{\alpha-1}}{\mu}} + \varphi_{\alpha-1} s_1 \cdot s_1^{\frac{m^{\alpha-1}}{\mu}} + \ldots + \varphi_{\alpha-1} s_{\nu-1} \cdot s_{\nu-1}^{\frac{m^{\alpha-1}}{\mu}}, \end{aligned}$$

where the φ_j are rational functions and

$$\begin{aligned} s^{\frac{1}{\mu}} &= A a^{\frac{1}{\mu}} a_1^{\frac{m^{\alpha}}{\mu}} a_2^{\frac{m^{2\alpha}}{\mu}} \ldots a_{\nu-1}^{\frac{m^{(\nu-1)\alpha}}{\mu}}, \\ s_1^{\frac{1}{\mu}} &= A_1 a^{\frac{m^{\alpha}}{\mu}} a_1^{\frac{m^{2\alpha}}{\mu}} a_2^{\frac{m^{3\alpha}}{\mu}} \ldots a_{\nu-1}^{\frac{1}{\mu}}, \\ \ldots\ s_{\nu-1}^{\frac{1}{\mu}} &= A_{\nu-1} a^{\frac{m^{(\nu-1)\alpha}}{\mu}} a_1^{\frac{1}{\mu}} a_2^{\frac{m^{\alpha}}{\mu}} \ldots a_{\nu-1}^{\frac{m^{(\nu-2)\alpha}}{\mu}}, \end{aligned}$$

generalising the form communicated to Crelle in march 1826. Naturally $a, a_1, \ldots, a_{\nu-1}$ are roots of a cyclic equation of degree dividing $\mu - 1$, but Abel does not say anything about it, that part of the paper being almost reduced to computations. Kronecker (1853) rediscovered this result, and stated it more precisely, also studying the form of the roots of cyclic equations.

The last part of the paper contains computations to establish the second statement relative to the problem "To find the most general algebraic expression which may satisfy an equation of given degree"; Sylow gives an interpretation of these computations at the end of the volume (p. 336–337).

4 Hyperelliptic Integrals

Abel studied Legendre's *Exercices de Calcul intégral* at the fall of 1823 and this book inspired him a series of new important discoveries; we already saw some of them. A memoir presented in 1826 to the Royal Society of Sciences in Throndhjem (*Œuvres*, t. I, p. 40–60) is devoted to a generalisation of Legendre's formula for the exchange of the parameter and the argument in elliptic integrals of the third kind. Abel considers an integral $p = \int \frac{e^{fx}\varphi x dx}{x-a}$ taken from $x = c$, where f is a rational function and $\varphi x = k(x+\alpha)^{\beta}(x+\alpha')^{\beta'} \dots (x+\alpha^{(n)})^{\beta^{(n)}}$ with $\beta, \beta', \dots$ rational numbers; derivating with respect to the parameter a and comparing with the derivative of $\frac{e^{fx}\varphi x}{x-a}$ with respect to x, he obtains

$$\frac{dp}{da} - \left(f'a + \frac{\varphi' a}{\varphi a}\right) p = \tag{38}$$
$$-\frac{e^{fx}\varphi x}{x-a} + \frac{e^{fc}\varphi c}{c-a} + \sum\sum p\gamma^{(p)} a^{p'} \int e^{fx}\varphi x \cdot x^{p-p'-2} dx$$
$$-\sum \frac{\beta^{(p)}}{a+\alpha^{(p)}} \int \frac{e^{fx}\varphi x dx}{x+\alpha^{(p)}} + \sum\sum \frac{\mu^{(p)}\delta^{(p)}}{(a+\varepsilon^{(p)})^{\mu^{(p)}-p'+2}} \int \frac{e^{fx}\varphi x dx}{(x+\varepsilon^{(p)})^{p'}}$$

if $fx = \sum \gamma^{(p)} x^p + \sum \frac{\delta^{(p)}}{(x+\varepsilon^{(p)})^{\mu^{(p)}}}$. When f is polynomial ($\delta^{(p)} = 0$) and

$$\psi x = (x+\alpha)(x+\alpha')\dots(x+\alpha^{(n)}),$$

there is another formula

$$\frac{dp}{da} - \left(f'a + \frac{\varphi' a}{\varphi a}\right) p = \frac{e^{fx}\varphi x \cdot \psi x}{\psi a(a-x)} - \frac{e^{fc}\varphi c \cdot \psi c}{\psi a(a-c)} \tag{39}$$
$$+ \sum\sum \varphi(p, p') \frac{a^{p'}}{\psi a} \int e^{fx}\varphi x \cdot x^p dx,$$

where $\varphi(p, p') = \frac{(p+1)\psi^{(p+p'+2)}}{2\cdot 3 \dots (p+p'+2)} + \frac{\left(\psi \frac{\varphi'}{\varphi} + f'\right)^{(p+p'+1)}}{2\cdot 3 \dots (p+p'+1)}$ ($F, F', \dots$ denoting the values at $x = 0$ of the successive derivatives of a function Fx).

As $\int \left(dp - \left(\frac{\varphi' a}{\varphi a} + f'a\right) pda\right) \frac{e^{-fa}}{\varphi a} = \frac{pe^{-fa}}{\varphi a}$, taking c such that $e^{fc}\varphi c = 0$ in (38) or such that $e^{fc}\varphi c \cdot \psi c = 0$ in (39), Abel gets

$$\frac{e^{-fa}}{\varphi a} \int \frac{e^{fx}\varphi x dx}{x-a} - e^{fx}\varphi x \int \frac{e^{-fa} da}{(a-x)\varphi a} \tag{40}$$
$$= \sum\sum p\gamma^{(p)} \int \frac{e^{-fa} a^{p'} da}{\varphi a} \cdot \int e^{fx}\varphi x \cdot x^{p-p'-2} dx$$
$$-\sum \beta^{(p)} \int \frac{e^{-fa} da}{(a+\alpha^{(p)})\varphi a} \cdot \int \frac{e^{fx}\varphi x dx}{x+\alpha^{(p)}}$$
$$+\sum\sum \mu^{(p)}\delta^{(p)} \int \frac{e^{-fa} da}{(a+\varepsilon^{(p)})^{\mu^{(p)}-p'+2}\varphi a} \cdot \int \frac{e^{fx}\varphi x dx}{(x+\varepsilon^{(p)})^{p'}}$$

and

$$\frac{e^{-fa}}{\varphi a}\int\frac{e^{fx}\varphi x.dx}{x-a}-e^{fx}\varphi x\cdot\psi x\int\frac{e^{-fa}da}{(a-x)\varphi a\cdot\psi a} \tag{41}$$
$$=\sum\sum\varphi(p,p')\int\frac{e^{-fa}a^{p'}da}{\varphi a\cdot\psi a}\cdot\int e^{fx}\varphi x\cdot x^p dx$$

when f is a polynomial; the integrals with respect to a must be taken from a value which annihilates $\frac{e^{-fa}}{\varphi a}$.

Abel gives special cases of these formulae, for instance when φ is the constant 1; if more-over $fx = x^n$, one has

$$e^{-a^n}\int\frac{e^{x^n}dx}{x-a}-e^{x^n}\int\frac{e^{-a^n}da}{a-x}$$
$$=n\left(\int e^{-a^n}a^{n-2}da\cdot\int e^{x^n}dx+\int e^{-a^n}a^{n-3}da\cdot\int e^{x^n}xdx+\dots\right.$$
$$\left.+\int e^{a^n}da\cdot\int e^{x^n}x^{n-2}dx\right).$$

When $fx = 0$, (40) gives

$$\varphi x\int\frac{da}{(a-x)\varphi a}-\frac{1}{\varphi a}\int\frac{\varphi xdx}{x-a}$$
$$=\beta\int\frac{da}{(a+\alpha)\varphi a}\cdot\int\frac{\varphi xdx}{x+\alpha}+\beta'\int\frac{da}{(a+\alpha')\varphi a}\cdot\int\frac{\varphi xdx}{x+\alpha'}+\dots$$
$$+\beta^{(n)}\int\frac{da}{(a+\alpha^{(n)})\varphi a}\cdot\int\frac{\varphi xdx}{x+\alpha^{(n)}}$$

and (41) $\frac{1}{\varphi a}\int\frac{\varphi xdx}{x-a}-\varphi x\cdot\psi x\int\frac{da}{(a-x)\varphi a\cdot\psi a}=\sum\sum\varphi(p,p')\int\frac{a^{p'}da}{\varphi a\cdot\psi a}\cdot\int\varphi x\cdot x^p dx$. If, in this last formula, $\beta=\beta'=\dots=\beta^{(n)}=m$, as $\varphi x=(\psi x)^m$, $\varphi(p,p')=(p+1+m(p+p'+2))k^{(p+p'+2)}$, where $k^{(j)}$ is the coefficient of x^j in ψx, so

$$\frac{1}{(\psi a)^m}\int\frac{(\psi x)^m dx}{x-a}-(\psi x)^{m+1}\int\frac{da}{(a-x)(\psi a)^{m+1}}$$
$$=\sum\sum k^{(p+p'+2)}(p+1+m(p+p'+2))\int\frac{a^{p'}da}{(\psi a)^{m+1}}\cdot\int(\psi x)^m x^p dx,$$

equality which reduces to

$$\sqrt{\psi a}\int\frac{dx}{(x-a)\sqrt{\psi x}}-\sqrt{\psi x}\int\frac{da}{(a-x)\sqrt{\psi a}} \tag{42}$$
$$=\frac{1}{2}\sum\sum(p-p')k^{(p+p'+2)}\int\frac{a^{p'}da}{\sqrt{\psi a}}\cdot\int\frac{x^p dx}{\sqrt{\psi x}}$$

when $m=-\frac{1}{2}$ and this gives an extension to hyperelliptic integrals of Legendre's formula. If, for example $\psi x = 1+\alpha x^n$, one has $\sqrt{1+\alpha a^n}\int\frac{dx}{(x-a)\sqrt{1+\alpha x^n}}$

$-\sqrt{1+\alpha x^n}\int\frac{da}{(a-x)\sqrt{1+\alpha a^n}} = \frac{\alpha}{2}\sum(n-2p'-2)\int\frac{a^{p'}da}{\sqrt{1+\alpha a^n}}\cdot\int\frac{x^{n-p'-2}dx}{\sqrt{1+\alpha x^n}}$. The elliptic case corresponds to $\psi x = (1-x^2)(1-\alpha x^2)$ and leads to

$$\begin{aligned}&\sqrt{(1-a^2)(1-\alpha a^2)}\int\frac{dx}{(x+a)\sqrt{(1-x^2)(1-\alpha x^2)}}\\&-\sqrt{(1-x^2)(1-\alpha x^2)}\int\frac{da}{(a+x)\sqrt{(1-a^2)(1-\alpha a^2)}}\\=\ &\alpha\int\frac{da}{\sqrt{(1-a^2)(1-\alpha a^2)}}\cdot\int\frac{x^2dx}{\sqrt{(1-x^2)(1-\alpha x^2)}}\\&-\alpha\int\frac{a^2da}{\sqrt{(1-a^2)(1-\alpha a^2)}}\cdot\int\frac{dx}{\sqrt{(1-x^2)(1-\alpha x^2)}}\end{aligned}$$

or, with $x=\sin\varphi$ and $a=\sin\psi$,

$$\begin{aligned}&\cos\psi\sqrt{1-\alpha\sin^2\psi}\int\frac{d\varphi}{(\sin\varphi+\sin\psi)\sqrt{1-\alpha\sin^2\varphi}}\\&-\cos\varphi\sqrt{1-\alpha\sin^2\varphi}\int\frac{d\psi}{(\sin\psi+\sin\varphi)\sqrt{1-\alpha\sin^2\varphi}}\\=\ &\alpha\int\frac{d\psi}{\sqrt{1-\alpha\sin^2\psi}}\cdot\int\frac{\sin^2\varphi d\varphi}{\sqrt{1-\alpha\sin^2\varphi}}\\&-\alpha\int\frac{\sin^2\psi d\psi}{\sqrt{1-\alpha\sin^2\psi}}\cdot\int\frac{d\varphi}{\sqrt{1-\alpha\sin^2\varphi}}.\end{aligned}$$

The formula (40) with $fx = x$ gives

$$\frac{e^{-a}}{\varphi a}\int\frac{e^x\varphi x dx}{x-a} - e^x\varphi x\int\frac{e^{-a}da}{(a-x)\varphi a} = -\sum\beta^{(p)}\int\frac{e^{-a}da}{(a+\alpha^{(p)})\varphi a}\cdot\int\frac{e^x\varphi x dx}{x+\alpha^{(p)}}$$

that is, for $\varphi x = \sqrt{x^2-1}$:

$$\begin{aligned}&e^x\sqrt{x^2-1}\int\frac{e^{-a}da}{(a-x)\sqrt{a^2-1}} - \frac{e^{-a}}{\sqrt{a^2-1}}\int\frac{e^x dx\sqrt{x^2-1}}{x-a}\\=\ &\frac{1}{2}\int\frac{e^{-a}da}{(a+1)\sqrt{a^2-1}}\cdot\int\frac{e^x dx\sqrt{x^2-1}}{x+1}\\&+\frac{1}{2}\int\frac{e^{-a}da}{(a-1)\sqrt{a^2-1}}\cdot\int\frac{e^x dx\sqrt{x^2-1}}{x-1}.\end{aligned}$$

Let us turn back to the formula (41) with $\beta=\beta'=\ldots=\beta^{(n)}=m$, but with f any polynomial:

$$\frac{e^{-fa}}{(\psi a)^m}\int\frac{e^{fx}(\psi x)^m dx}{x-a}-e^{fx}(\psi x)^{m+1}\int\frac{e^{-fa}da}{(a-x)(\psi a)^{m+1}}$$
$$=\sum\sum((p+p'+2)\gamma^{(p+p'+2)}+(p+1+m(p+p'+2))k^{(p+p'+2)})$$
$$\times\int\frac{e^{-fa}a^{p'}da}{(\psi a)^{m+1}}\cdot\int e^{fx}(\psi x)^m x^p dx,$$

that is

$$e^{-fa}\sqrt{\psi a}\int\frac{e^{fx}dx}{(x-a)\sqrt{\psi x}}-e^{fx}\sqrt{\psi x}\int\frac{e^{-fa}da}{(a-x)\sqrt{\psi a}}$$
$$=\sum\sum\left((p+p'+2)\gamma^{(p+p'+2)}+\frac{1}{2}(p-p')k^{(p+p'+2)}\right)$$
$$\times\int\frac{e^{-fa}a^{p'}da}{(\psi a)^{m+1}}\cdot\int e^{fx}(\psi x)^m x^p dx$$

when $m=-\frac{1}{2}$; if moreover $fx=x$ and $\psi x=1-x^2$, this gives $e^{-a}\sqrt{1-a^2}\times\int\frac{e^x dx}{(x-a)\sqrt{1-x^2}}=e^x\sqrt{1-x^2}\int\frac{e^{-a}da}{(a-x)\sqrt{1-a^2}}$ or $\cos\psi e^{\sin\psi}\int\frac{e^{\sin\varphi}d\varphi}{\sin\varphi+\sin\psi}=\cos\varphi e^{\sin\varphi}\times\int\frac{e^{\sin\psi}d\psi}{\sin\psi+\sin\varphi}$ (integrals from φ, $\psi=\frac{\pi}{2}$).

Abel also applies these formulae to definite integrals: the formula (40) with f polynomial gives

$$\int_{x'}^{x''}\frac{e^{fx}\varphi x dx}{x-a} \tag{43}$$
$$=e^{fa}\varphi a\sum\sum(p+p'+2)\gamma^{(p+p'+2)}\int_{a'}\frac{e^{-fa}a^{p'}da}{\varphi a}\cdot\int_{x'}^{x''}e^{fx}\varphi x dx$$
$$-e^{fa}\varphi a\sum\beta^{(p)}\int_{a'}\frac{e^{-fa}da}{(a+\alpha^{(p)})\varphi a}\cdot\int\frac{e^{fx}\varphi x dx}{x+\alpha^{(p)}}$$

when x', x'' annihilate $e^{fx}\varphi x$ and a' annihilates $\frac{e^{-fa}}{\varphi a}$. For $fx=0$, this gives

$$\int_{x'}^{x''}\frac{\varphi x dx}{x-a}=-\varphi a\sum\beta^{(p)}\int_{a'}\frac{da}{(a+\alpha^{(p)})\varphi a}\cdot\int_{x'}^{x''}\frac{\varphi x dx}{x+\alpha^{(p)}}$$

and for $\varphi x=1$, $\int_{x'}^{x''}\frac{e^{fx}dx}{x-a}=e^{fa}\sum\sum(p+p'+2)\gamma^{(p+p'+2)}\int_{a'}e^{-fa}a^{p'}da\cdot\int_{x'}^{x''}e^{fx}x^p dx$.

The formula (43), with $\frac{e^{-fa''}}{\varphi a''}=0$ gives

$$\sum \beta^{(p)} \int_{a'}^{a''} \frac{e^{-fa}da}{(a+\alpha^{(p)})\varphi a} \cdot \int_{x'}^{x''} \frac{e^{fx}\varphi x dx}{x+\alpha^{(p)}}$$
$$= \sum\sum (p+p'+2)\gamma^{(p+p'+2)} \int_{a'}^{a''} \frac{e^{-fa}a^{p'}da}{\varphi a} \cdot \int_{x'}^{x''} e^{fx}\varphi x dx,$$

for instance $\sum \beta^{(p)} \int_{a'}^{a''} \frac{e^{-ka}da}{(a+\alpha^{(p)})\varphi a} \cdot \int_{x'}^{x''} \frac{e^{kx}\varphi x dx}{x+\alpha^{(p)}} = 0$ for $fx = kx$ and $\varphi x = 1$,

$$\sum\sum (p+p'+2)\gamma^{(p+p'+2)} \int_{a'}^{a''} e^{-fa}a^{p'}da \cdot \int_{x'}^{x''} e^{fx}x^p dx = 0.$$

Using now (41) with x', x'' annihilating $e^{fx}\varphi x \cdot \psi x$, Abel gets

$$\int_{x'}^{x''} \frac{e^{fx}\varphi x dx}{x-a} = e^{fa}\varphi a \sum\sum \varphi(p, p') \int_{a'} \frac{e^{-fa}a^{p'}da}{\varphi a \cdot \psi a} \cdot \int_{x'}^{x''} e^{fx}\varphi x \cdot x^p dx$$

and, when $\beta = \beta' = \ldots = \beta^{(n)} = m$,

$$\int_{x'}^{x''} \frac{e^{fx}(\psi x)^m dx}{x-a} = e^{fa}(\psi a)^m \sum\sum \varphi(p, p') \int_{a'} \frac{e^{-fa}a^{p'}da}{(\psi a)^{m+1}} \cdot \int_{x'}^{x''} e^{fx}(\psi x)^m x^p dx. \tag{44}$$

For $fx = 0$ and $m = -\frac{1}{2}$, this gives

$$\int_{x'}^{x''} \frac{dx}{(x-a)\sqrt{\psi x}} = \frac{1}{2\sqrt{\psi a}} \sum\sum (p-p')k^{(p+p'+2)} \int_{a'} \frac{a^{p'}da}{\sqrt{\psi a}} \cdot \int_{x'}^{x''} \frac{x^p dx}{\sqrt{\psi x}}$$

expressing the periods of an hyperelliptic integral of the third kind by means of the periods of the integrals of the first two kinds; in the elliptic case,

$$\psi x = (1-x^2)(1-\alpha x^2),$$

one has

$$\sqrt{(1-a^2)(1-\alpha a^2)}\int_{x'}^{x''}\frac{dx}{(x-a)\sqrt{(1-x^2)(1-\alpha x^2)}}$$

$$=\alpha\int_{a'}\frac{da}{\sqrt{(1-a^2)(1-\alpha a^2)}}\cdot\int_{x'}^{x''}\frac{x^2dx}{\sqrt{(1-x^2)(1-\alpha x^2)}}$$

$$-\alpha\int_{a'}\frac{a^2da}{\sqrt{(1-a^2)(1-\alpha a^2)}}\cdot\int_{x'}^{x''}\frac{dx}{\sqrt{(1-x^2)(1-\alpha x^2)}}$$

with x', x'' and $a' = \pm 1$ or $\pm\sqrt{\frac{1}{\alpha}}$.

From (44) with $\psi x = 1 - x^{2n}$, $x' = -1$, $x = 1$ and $a' = 1$, Abel deduces

$$\int_{-1}^{1}\frac{dx}{(x-a)(1-x^{2n})^m}$$

$$=\frac{\Gamma(-m+1)}{n(1-a^{2n})^m}\sum(2p+1-2mn)\frac{\Gamma\left(\frac{1+2p}{2n}\right)}{\Gamma\left(-m+1+\frac{1+2p}{2n}\right)}\int_1\frac{a^{2n-2p-2}da}{(1-a^{2n})^{1-m}}$$

for $m > 0$. If, for example $m = \frac{1}{2}$ and $n = 3$, this gives

$$\int_{-1}^{1}\frac{dx}{(x-a)\sqrt{1-x^6}}=-\frac{2}{3}\frac{\sqrt{\pi}}{\sqrt{1-a^6}}\frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{2}{3}\right)}\int_1\frac{a^4da}{\sqrt{1-a^6}}$$

$$+\frac{2}{3}\frac{\sqrt{\pi}}{\sqrt{1-a^6}}\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{3}\right)}\int_1\frac{da}{\sqrt{1-a^6}}.$$

Now if $\frac{e^{-fa''}}{\varphi a''} = 0$, $\sum\sum\varphi(p,p')\int_{a'}^{a''}\frac{e^{-fa}a^{p'}da}{\varphi a\cdot\psi a}\cdot\int_{x'}^{x''}e^{fx}\varphi x\cdot x^p dx = 0$; for instance, when $\varphi x = (x+\alpha)^{\beta}(x+\alpha')^{\beta'}\dots(x+\alpha^{(n)})^{\beta^{(n)}}$ and $\psi a = (a+\alpha)^{1-\beta}(a+\alpha')^{1-\beta'}\dots$ $(a+\alpha^{(n)})^{1-\beta^{(n)}}$, with $-1 < \beta, \beta', \dots, \beta^{(n)} < 0$, one has

$$\sum\sum\varphi(p,p')\int_{a'}^{a''}\frac{a^{p'}da}{\psi a}\cdot\int_{x'}^{x''}\frac{x^p dx}{\varphi x} = 0$$

if $x' = -\alpha^{(p)}$, $x'' = -\alpha^{(p')}$, $a' = -\alpha^{(q)}$ and $a'' = -\alpha^{(q')}$. When $\beta = \beta' = \dots = \beta^{(n)} = \frac{1}{2}$, denoting by φx a polynomial of degree n, with roots $\alpha, \alpha', \alpha'', \dots$ and by $F(p,x)$ the integral $\int\frac{x^p dx}{\sqrt{\varphi x}}$, this relation becomes

$$\begin{aligned}
&\sum\sum (p-p')k^{(p+p'+2)}F(p,\alpha)F(p,',\alpha') \\
&\quad+\sum\sum (p-p')k^{(p+p'+2)}F(p,\alpha'')F(p',\alpha''') \\
&=\sum\sum (p-p')k^{(p+p'+2)}F(p,\alpha)F(p,',\alpha''') \\
&\quad+\sum\sum (p-p')k^{(p+p'+2)}F(p,\alpha'')F(p',\alpha'),
\end{aligned}$$

and, $\varphi x = (1-x^2)(1-c^2x^2)$ (elliptic case), $\alpha = 1$, $\alpha' = -1$, $\alpha'' = \frac{1}{c}$ and $\alpha''' = -\frac{1}{c}$, we find

$$F(1)E\left(\frac{1}{c}\right) = E(1)F\left(\frac{1}{c}\right),$$

where $Fx = \int\limits_0 \frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$ and $Ex = \int\limits_0 \frac{x^2dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$.

A short posthumous paper *Sur une propriété remarquable d'une classe très étendue de fonctions transcendantes* (*Œuvres*, t. II, p. 43–46, mem. VIII) contains some of the same formulae. Abel starts from a differential equation $0 = sy + t\frac{dy}{dx}$, where $t = \varphi x$ and $s = fx$ are polynomials; then, for $r = \frac{\varphi' x - fx}{x-a} - \frac{\varphi x}{(x-a)^2} = -\frac{\varphi a}{(x-a)^2} - \frac{fa}{x-a} + R$, with

$$\begin{aligned}
R = \frac{1}{2}\varphi'' a - f'a + \left(\frac{1}{3}\varphi''' a - \frac{1}{2}f''a\right)(x-a) \\
+\left(\frac{1}{2\cdot 4}\varphi'''' a - \frac{1}{2\cdot 3}f'''a\right)(x-a)^2 + \dots,
\end{aligned}$$

and $y = \psi x$ solution of the differential equation, one has

$$\int rydx = \frac{y\varphi x}{x-a} \quad \text{or} \quad \frac{y\varphi x}{x-a} = -\varphi a\frac{dz}{da} - fa\cdot z + \int Rydx,$$

where $z = \int \frac{ydx}{x-a}$. If $z = p\psi a$, this equality becomes $\int Rydx - \frac{\varphi x\cdot\psi x}{x-a} = \varphi a\cdot\psi a\frac{dp}{da}$ and we have

$$\begin{aligned}
p = \frac{1}{\psi a}\int\frac{\psi x dx}{x-a} - \psi x\cdot\varphi x\int\frac{da}{(a-x)\psi a\cdot\varphi a} = \iint\frac{R\cdot\psi x}{\varphi a\cdot\psi a}dxda \\
= \sum((n+1)\alpha_{m+n+2} - \beta_{m+n+1})\int\frac{a^m da}{\varphi a\cdot\psi a}\int x^n\psi x dx,
\end{aligned}$$

where α_k (resp. β_k) is the coefficient of x^k in φx (resp. fx); the origin of integration in x (resp. in a) must annihilate $\psi x\cdot\varphi x$ (resp. $\frac{1}{\psi a}$). Note that, up to a constant factor, $\psi x = e^{-\int\frac{fx}{\varphi x}dx}$ has the form $\frac{e^p}{(x-\delta)^m(x-\delta_1)^{m_1}\dots}$, where p is a rational function (not the preceding p !) and $0 < m, m_1, \dots < 1$. When $\psi x = \frac{1}{\sqrt{\varphi x}}$, $fx - \frac{1}{2}\varphi' x = 0$ and $\beta_m = \frac{1}{2}(m+1)\alpha_{m+1}$ and we find the formula (42).

The following paper in the second volume of Abel's *Works* (p. 47–54, mem. IX) extends this theory to the case of a linear differential equation of order m with polynomial coefficients:

$$0 = sy + s_1 \frac{dy}{dx} + \ldots + s_m \frac{d^m y}{dx^m}, \tag{45}$$

$s_k = \varphi_k(x)$ polynomial. Abel looks for a function r such that

$$\int rydx = vy + v_1 \frac{dy}{dx} + v_{m-2} \frac{d^{m-2}y}{dx^{m-2}} + ts_m \frac{d^{m-1}y}{dx^{m-1}},$$

with t a given function, taken equal to $\frac{1}{x-a}$. He finds that $v_{\mu-1} = s_\mu t - \frac{dv_\mu}{dx}$ for $0 \leq \mu \leq m-1$ (with $v_{-1} = -r$ and $v_{m-1} = ts_m$), so that $v_{\mu-1} = s_\mu t - \frac{d(s_{\mu+1}t)}{dx} + \frac{d^2(s_{\mu+2}t)}{dx^2} - \ldots$ and

$$-r = st - \frac{d(s_1 t)}{dx} + \frac{d^2(s_2 t)}{dx^2} - \ldots + (-1)^m \frac{d^m(s_m t)}{dx^m}.$$

Now $s_\mu t = \frac{\varphi_\mu(a)}{x-a} + R_\mu$, where R_μ is polynomial in x, so

$$-r = \frac{\varphi a}{x-a} + \frac{\varphi_1 a}{(x-a)^2} + \Gamma(3)\frac{\varphi_2 a}{(x-a)^3} + \ldots + \Gamma(m+1)\frac{\varphi_m a}{(x-a)^{m+1}} + \rho,$$

with $\rho = R - \frac{dR_1}{dx} + \ldots + (-1)^m \frac{d^m R}{dx^m}$. Thus the integral $z = \int \frac{ydx}{x-a}$ satisfies a differential equation $\varphi a \cdot z + \varphi_1 a \cdot \frac{dz}{da} + \ldots + \varphi_m a \cdot \frac{d^m z}{da^m} = -\chi' - \int \rho ydx$, where $\chi = vy + v_1 \frac{dy}{dx} + v_{m-2} \frac{d^{m-2}y}{dx^{m-2}} + ts_m \frac{d^{m-1}y}{dx^{m-1}}$ and $\chi' = \chi - \chi_0$ (where χ_0 is the value of χ at the origin of integration); Abel writes the solution of this equation in terms of a fundamental system of solutions of (45).

In the second part of the paper, Abel wishes to find coefficients $\alpha_1, \alpha_2, \ldots, \alpha_m$ depending on a such that $z = \int \left(\frac{\alpha_1}{x-a} + \frac{\alpha_2}{(x-a)^2} + \ldots + \frac{\Gamma m \cdot \alpha_m}{(x-a)^m}\right) ydx$ satisfy a differential equation of the form $\beta z + \gamma \frac{dz}{da} = \chi + \int \rho ydx$; he writes induction relations between the α_μ and, supposing $-\frac{\beta}{\gamma} = \varepsilon$ constant, a differential equation to determine γ.

The first article on Abelian integrals published by Abel in *Crelle's Journal* (1826, *Œuvres*, t. I, p. 104–144) is devoted to the search for differential forms $\frac{\rho dx}{\sqrt{R}}$, with ρ and R polynomials, such that their integrals have the form $\log \frac{p+q\sqrt{R}}{p-q\sqrt{R}} = z$, with p, q polynomials. As $dz = \frac{pqdR+2(pdq-qdp)R}{(p^2-q^2R)\sqrt{R}}$, one must have $\rho = \frac{M}{N}$ with $M = pq\frac{dR}{dx} + 2\left(p\frac{dq}{dx} - q\frac{dp}{dx}\right)R$ and $N = p^2 - q^2 R$. It follows that $q\rho = 2\frac{dp}{dx} - p\frac{dN}{Ndx}$ and $p\frac{dN}{Ndx}$ must be polynomials. If

$$N = (x+a)^m (x+a_1)^{m_1} \ldots (x+a_n)^{m_n},$$

$\frac{dN}{Ndx} = \frac{m}{x+a} + \frac{m_1}{x+a_1} + \ldots + \frac{m_n}{x+a_n}$ and we see that p is divisible by

$$(x+a)(x+a_1)\cdots(x+a_n).$$

Hence $p = (x + a)(x + a_1) \cdots (x + a_n) p_1$ and

$$(x + a)^m (x + a_1)^{m_1} \cdots (x + a_n)^{m_n} = p_1^2 (x + a)^2 (x + a_1)^2 \cdots (x + a_n)^2 - q^2 R$$

which shows that $m = m_1 = \ldots = m_n = 1$ if p and q are relatively prime (R is supposed to be square free) and that $R = (x + a)(x + a_1) \cdots (x + a_n) R_1 = N R_1$. Now p_1, q, N and R_1 are determined by the equation $p_1^2 N - q^2 R_1 = 1$ and so $\rho = p_1 q \frac{dR}{dx} + 2\left(p\frac{dq}{dx} - q\frac{dp}{dx}\right) R_1$.

Abel studies $p_1^2 N - q^2 R_1 = 1$, or more generally $p_1^2 N - q^2 R_1 = v$ (where v is polynomial of degree less than the mean of the degrees of N and R_1), as a Diophantine equation in the ring of polynomials. The first observation is that Np_1^2 and $R_1 q^2$ must have the same degree and so $\delta R = \delta(NR_1) = 2(\delta q - \delta p_1 + \delta R_1)$ (where δ designates the degree) is an even number, so we can put $\delta N = n - m$ and $\delta R_1 = n + m$. By Euclidian division $R_1 = Nt + t'$, with $\delta t = 2m$ and $\delta t' < n - m$. Using the method of indeterminate coefficients, Abel shows that there exist t_1 and t_1' such that $\delta t_1' < m$ and $t = t_1^2 + t_1'$. The equation now becomes $(p_1^2 - q^2 t_1^2) N - q^2 s = v$, with $s = N t_1' + t'$ and shows that $\left(\frac{p_1}{q}\right)^2$ and t_1^2 differ by a rational fraction of degree less than δt_1, and as a consequence the polynomial part of $\frac{p_1}{q}$ is $\pm t_1$, say t_1. Thus $p_1 = t_1 q + \beta$ with $\delta\beta < \delta q$ and one can verify that $\left(\frac{q}{\beta} - \frac{t_1 N}{s}\right)^2 = \frac{R_1 N}{s^2} - \frac{v}{s\beta^2}$; if $R = R_1 N = r^2 + r'$ with $\delta r' < \delta r$, we see as before that $\frac{q}{\beta} - \frac{t_1 N}{s}$ and $\frac{r}{s}$ have the same polynomial part, so $\frac{q}{\beta}$ and $\frac{r + t_1 N}{s}$ have the same polynomial part 2μ and $q = 2\mu\beta + \beta_1$ with $\delta\beta_1 < \delta\beta$. Now the Diophantine equation becomes

$$s_1 \beta^2 - 2 r_1 \beta \beta_1 - s \beta_1^2 = v$$

where $s_1 = N + 4\mu t_1 N - 4 s \mu^2$ and $r_1 = 2\mu s - t_1 N$; as $r_1^2 + s s_1 = R$, $\delta r_1 = n$ and $\delta s, \delta s_1 < n$.

Our equation may be written $\left(\frac{\beta}{\beta_1} - \frac{r_1}{s_1}\right)^2 = \left(\frac{r_1}{s_1}\right)^2 + \frac{s}{s_1} + \frac{v}{s_1 \beta_1^2}$ which shows that $\frac{\beta}{\beta_1} - \frac{r_1}{s_1}$ and $\frac{r_1}{s_1}$ have the same polynomial part up to sign. Writing μ_1 for the polynomial part of $\frac{r_1}{s_1}$, the polynomial part of $\frac{\beta}{\beta_1}$ is $2\mu_1$ and $\beta = 2\beta_1\mu_1 + \beta_2$ with $\delta\beta_2 < \delta\beta_1$. The equation can now be rewritten $s_2 \beta_1^2 - 2 r_2 \beta_1 \beta_2 - s_1 \beta_2^2 = -v$, where $r_2 = 2\mu_1 s_1 - r_1$ and $s_2 = s + 4 r_1 \mu_1 - 4 s_1 \mu_1^2$, from which it is easy to see that $\delta r_2 = \delta r_1 > \delta s_2$. Continuing in this way, we obtain

$$s_n \beta_{n-1}^2 - 2 r_n \beta_{n-1} \beta_n - s_{n-1} \beta_n^2 = (-1)^{n-1} v$$

with $\beta_{n-1} = 2\mu_n \beta_n + \beta_{n+1}$, μ_n integral part of $\frac{r_n}{s_n}$, $r_n = 2\mu_{n-1} s_{n-1} - r_{n-1}$ and $s_n = s_{n-2} + 4 r_{n-1} \mu_{n-1} - 4 s_{n-1} \mu_{n-1}^2$; as $\delta\beta > \delta\beta_1 > \ldots > \delta\beta_n > \ldots$, there is an m such that $\beta_m = 0$, giving the equation $s_m \beta_{m-1}^2 = (-1)^{m-1} v$. The sequence (β_k) is determined by the Euclidian algorithm and β_{m-1} is the g.c.d. of β and β_1, and so 1 if p and q are relatively prime and $v = (-1)^{m-1} s_m$.

In the initial problem, we had $v = 1$, so we must take s_m constant. As the s_k are of degree k, this gives $n - 1$ conditions on the $2n$ coefficients of R_1 and N, once the index m is chosen. Thus $\frac{p_1}{q} = t_1 + \cfrac{1}{2\mu + \cfrac{1}{2\mu_1 + \ddots + \cfrac{1}{2\mu_{m-1}}}}$ and $\rho = \left(p_1\frac{dN}{dx} + 2N\frac{dp_1}{dx}\right) : q$, of degree $n-1$.

Abel adds some observations. The first is that one may put $N = 1$ in the problem, for

$$\log\frac{p+q\sqrt{R}}{p-q\sqrt{R}} = \frac{1}{2}\log\frac{p_1^2N + q^2R_1 + 2p_1q\sqrt{R}}{p_1^2N + q^2R_1 - 2p_1q\sqrt{R}} = \frac{1}{2}\log\frac{p' + q'\sqrt{R}}{p' - q'\sqrt{R}}$$

with $p' = p_1^2N + q^2R_1$ and $q' = 2p_1q$. In that case, the equation takes the Pell-Fermat form $p'^2 - q'^2R = 1$ and $\rho = \frac{2}{q'}\frac{dp'}{dx}$. The second is that, if $\frac{\alpha_k}{\beta_k}$ is the k-th convergent of the continued fraction for $\frac{p_1}{q}$, $\alpha_k^2N - \beta_k^2R_1 = (-1)^{k-1}s_k$ and so, putting

$$z_k = \alpha_k\sqrt{N} + \beta_k\sqrt{R_1} \quad \text{and} \quad z_k' = \alpha_k\sqrt{N} - \beta_k\sqrt{R_1},$$

we get $\frac{z_k}{z_k'} = \frac{r_k+\sqrt{R}}{r_k-\sqrt{R}}\frac{z_{k-1}}{z_{k-1}'}$. Lastly,

$$\log\frac{p_1\sqrt{N} + q\sqrt{R_1}}{p_1\sqrt{N} - q\sqrt{R_1}} = \log\frac{t_1\sqrt{N} + \sqrt{R_1}}{t_1\sqrt{N} - \sqrt{R_1}} + \log\frac{r_1\sqrt{N} + \sqrt{R}}{r_1\sqrt{N} - \sqrt{R}} + \ldots + \log\frac{r_m\sqrt{N} + \sqrt{R}}{r_m\sqrt{N} - \sqrt{R}}.$$

Abel also shows that

$$\rho dx = 2(\frac{1}{2}t_1dN + Ndt_1 + dr_1 + \ldots + dr_m - \mu ds - \ldots - \mu_{m-1}ds_{m-1}).$$

As $\left(\frac{p_1}{q}\right)^2 = \frac{R_1}{N} + \frac{v}{q^2N}$, one sees that the continued fraction for $\frac{p_1}{q}$ is obtained from that for $\sqrt{\frac{R_1}{N}}$. Let us suppose that $N = 1$, so that $t_1 = r$ and $\sqrt{R} = r + \cfrac{1}{2\mu + \cfrac{1}{2\mu_1} + \ddots}$. An easy computation shows that $r_m^2 + s_ms_{m-1} = r_{m-1}^2 + s_{m-1}s_{m-2} = \ldots = r_1^2 + ss_1 = R = r^2 + s$. If $s_m = a$ constant, we have $r_{m+1}^2 - r^2 = s - as_{m+1}$ and, since $\delta r_{m+1} = \delta r > \delta s, \delta s_{m+1}$, this implies $r_{m+1} = r$ and $s_{m+1} = \frac{s}{a}$. This shows that the polynomial part μ_{m+1} of $\frac{r_{m+1}}{s_{m+1}}$ is equal to $a\mu$ and $s_{m+2} = as_1$, $r_{m+2} = r_1$ and so on, in general we have $r_{m+n} = r_{n-1}, r_{m-n} = r_n, s_{m\pm n} = a^{(-1)^n}s_{n-1}$ and $\mu_{m\pm n} = a^{(-1)^{n-1}}\mu_{n-1}$. This shows that the continued fraction is periodic with partial quotients $r, 2\mu, 2\mu_1, \ldots, \frac{2\mu_1}{a}, 2a\mu, \frac{2r}{a}, 2a\mu, \frac{2\mu_1}{a}, \ldots, 2\mu, 2r, 2\mu, \ldots$; if $m = 2k-1$ is odd, $s_{k-1} = a^{(-1)^k}s_{k-1}$ and $a = 1$. Conversely, one sees that if $\sqrt{R}$ has a continued fraction of the preceding form, $s_m = a$. Abel may conclude by a criterion for the existence of a polynomial ρ such that $\int\frac{\rho dx}{\sqrt{R}} = \log\frac{y+\sqrt{R}}{y-\sqrt{R}}$ with y rational: $\sqrt{R}$ must be of the form $r + \cfrac{1}{2\mu + \cfrac{1}{2\mu_1 + \ddots + \cfrac{1}{2\mu_1 + \cfrac{1}{2\mu + \cfrac{1}{2r + \cfrac{1}{2\mu + \cfrac{1}{2\mu_1 + \ddots}}}}}}}$

$$\text{and then} \quad y = r + \cfrac{1}{2\mu + \cfrac{1}{2\mu_1 + \ddots + \cfrac{1}{2\mu_1 + \cfrac{1}{2\mu + \frac{1}{2r}}}}}.$$

Conversely, Abel proves that $s_k = s_{k-2}$ implies $r_k = r_{k-1}$, $\mu_k = \mu_{k-2}$, $r_{k+1} = r_{k-2}$, $s_{k+1} = s_{k-3}$ etc. and generally $r_{k+n} = r_{k-n-1}$, $\mu_{k+n} = \mu_{k-n-2}$, $s_{k+n} = s_{k-n-2}$; for $n = k-1$, this gives $s_{2k-1} = s_{-1} = 1$. If $s_k = cs_{k-1}$, he finds $\mu_k = \frac{1}{c}\mu_{k-1}$, $r_{k+1} = r_{k-1}$, $s_{k+1} = \frac{1}{c}s_{k-2}, \dots, s_{2k} = c^{(-1)^k}$. The integral found is

$$\int \frac{2}{\sqrt{R}}(dr + dr_1 + \ldots + dr_{k-1} + \frac{1}{2}dr_k - \mu ds - \mu_1 ds_1 - \ldots - \mu_{k-1}ds_{k-1})$$
$$= \log\frac{r+\sqrt{R}}{r-\sqrt{R}} + \log\frac{r_1+\sqrt{R}}{r_1-\sqrt{R}} + \ldots + \log\frac{r_{k-1}+\sqrt{R}}{r_{k-1}-\sqrt{R}} + \frac{1}{2}\log\frac{r_k+\sqrt{R}}{r_k-\sqrt{R}}$$

when $m = 2k$ and

$$\int \frac{2}{\sqrt{R}}(dr + dr_1 + \ldots + dr_{k-1} - \mu ds - \mu_1 ds_1 - \ldots - \mu_{k-2}ds_{k-2} - \frac{1}{2}\mu_{k-1}ds_{k-1})$$
$$= \log\frac{r+\sqrt{R}}{r-\sqrt{R}} + \log\frac{r_1+\sqrt{R}}{r_1-\sqrt{R}} + \ldots + \log\frac{r_{k-1}+\sqrt{R}}{r_{k-1}-\sqrt{R}}$$

when $m = 2k-1$.

In the elliptic case, where R is of degree $4 = 2n$, $n = 2$ and the s_m are of degree 1, so there is only one condition to write in order to have $s_m =$ const. If $R = (x^2+ax+b)^2 + ex$, we have $r = x^2+ax+b$, $s = ex$, $\mu = \frac{x+a}{e}$ and then $r_1 = x^2+ax-b$, $s_1 = \frac{4b}{e}x + \frac{4ab}{e} + 1$, $\mu_1 = \frac{e}{4b}x - \frac{e^2}{16b^2}$ and $s_2 = \left(\frac{ae^2}{4b^2} + \frac{e^3}{16b^3}\right)x - \frac{e^2}{4b^2}\left(\frac{ae}{4b} + \frac{e^2}{16b^2} - b\right)$. In order to make s_1 constant, we have only to put $b = 0$ and we find that $\int \frac{(3x+a)dx}{\sqrt{(x^2+ax)^2+ex}} = \log\frac{x^2+ax+\sqrt{R}}{x^2+ax-\sqrt{R}}$, with $R = (x^2+ax)^2+ex$. In order to make s_2 constant, we put $e = -4ab$, $R = (x^2+ax+b)^2 - 4abx$ and find

$$\int \frac{(4x+a)dx}{\sqrt{(x^2+ax+b)^2-4abx}} = \log\frac{x^2+ax+b+\sqrt{R}}{x^2+ax+b-\sqrt{R}} + \frac{1}{2}\log\frac{x^2+ax-b+\sqrt{R}}{x^2+ax-b-\sqrt{R}}.$$

Abel also computes the case in which s_3 is constant, which is given by

$$e = -2b(a \pm \sqrt{a^2+4b})$$

and

$$\int \frac{(5x+\frac{3}{2}\mp\frac{1}{2}\sqrt{a^2+4b})dx}{\sqrt{(x^2+ax+b)^2-2bx(a\pm\sqrt{a^2+4b})}} = \log\frac{x^2+ax+b+\sqrt{R}}{x^2+ax+b-\sqrt{R}} + \log\frac{x^2+ax-b+\sqrt{R}}{x^2+ax-b-\sqrt{R}} \quad ,$$

and the case where s_4 is constant, which leads to $e = -b(3a \pm \sqrt{a^2+8b})$ and

$$\int \frac{(6x+\frac{3}{2}a-\frac{1}{2}\sqrt{a^2+8b})dx}{\sqrt{(x^2+ax+b)^2-b(3a+\sqrt{a^2+8b})x}}$$
$$= \log\frac{x^2+ax+b+\sqrt{R}}{x^2+ax+b-\sqrt{R}} + \log\frac{x^2+ax-b+\sqrt{R}}{x^2+ax-b-\sqrt{R}}$$
$$+\frac{1}{2}\log\frac{x^2+ax+\frac{1}{4}a(a-\sqrt{a^2+8b})+\sqrt{R}}{x^2+ax+\frac{1}{4}a(a-\sqrt{a^2+8b})-\sqrt{R}}.$$

At the end of the memoir, Abel states a theorem according to which, whenever an integral $\int \frac{\rho dx}{\sqrt{R}}$, ρ and R polynomials, may be expressed by logarithms, it is always in the form $A\log\frac{p+q\sqrt{R}}{p-q\sqrt{R}}$, with A constant, p and q polynomials.

Chebyshev (1860) and Zolotarev (1872) studied the same problem in the elliptic case looking for arithmetical conditions on the coefficients of R, these latter supposed to be integers.

The first text written by Abel on elliptic functions (between 1823 and 1825), with the title *Théorie des transcendantes elliptiques* (*Œuvres*, t. II, p. 87–188), also deals with this problem but it was not published by Abel. In the first chapter, Abel studies the conditions under which an elliptic integral $\int \frac{Pdx}{\sqrt{R}}$, with P a rational function and $R = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4$, is an algebraic function. At first taking P polynomial, he observes that this algebraic function must be rational in x and $\sqrt{R}$, so of the form $Q' + Q\sqrt{R}$ with Q' and Q rational; since dQ' is rational, we may write $d(Q\sqrt{R}) = \frac{Pdx}{\sqrt{R}}$. The function Q is a polynomial otherwise its poles would remain as poles in the differential: $Q = f(0) + f(1)x + \ldots + f(n)x^n$ and $d(Q\sqrt{R}) = S\frac{dx}{\sqrt{R}}$, with

$$S = R\frac{dQ}{dx} + \frac{1}{2}Q\frac{dR}{dx} = \varphi(0) + \varphi(1)x + \ldots + \varphi(m)x^m.$$

This gives

$$\varphi(p) = (p+1)f(p+1)\ldots\alpha + \left(p+\frac{1}{2}\right)f(p)\cdot\beta + pf(p-1)\cdot\gamma$$
$$+\left(p-\frac{1}{2}\right)f(p-2)\cdot\delta + (p-1)f(p-3)\cdot\varepsilon$$

and $m = n+3$. Abel draws the conclusion that the integrals $\int \frac{x^m dx}{\sqrt{R}}$ may be expressed as linear combinations of those with $0 \le m \le 2$ and an algebraic function; but $\int \frac{dx}{\sqrt{R}}$, $\int \frac{xdx}{\sqrt{R}}$ and $\int \frac{x^2dx}{\sqrt{R}}$ are independent when the reductions admitted involve only algebraic functions. The reduction of $\int \frac{x^m dx}{\sqrt{R}}$ is given by a system of $m-2$ linear equations $\varphi(p) = 0$ for $3 \le p \le m-1$, $\varphi(m) = -1$ to determine the $f(p)$ $(0 \le p \le m-3)$ and the formulae

$$\varphi(0) = f(1)\cdot\alpha + \frac{1}{2}f(0)\cdot\beta, \quad \varphi(1) = 2f(2)\cdot\alpha + \frac{3}{2}f(1)\cdot\beta + f(0)\cdot\gamma,$$
$$\varphi(2) = 3f(3)\cdot\alpha + \frac{5}{2}f(2)\cdot\beta + 2f(1)\cdot\gamma + \frac{3}{2}f(0)\cdot\delta.$$

For instance

$$\int \frac{x^4dx}{\sqrt{R}} = \left(\frac{5}{24}\frac{\beta\delta}{\varepsilon^2} - \frac{1}{3}\frac{\alpha}{\varepsilon}\right)\int\frac{dx}{\sqrt{R}} + \left(\frac{5}{12}\frac{\gamma\delta}{\varepsilon^2} - \frac{1}{2}\frac{\beta}{\varepsilon}\right)\int\frac{xdx}{\sqrt{R}}$$
$$+\left(\frac{5}{8}\frac{\delta^2}{\varepsilon^2} - \frac{2}{3}\frac{\gamma}{\varepsilon}\right)\int\frac{x^2dx}{\sqrt{R}} - \left(\frac{5}{12}\frac{\delta}{\varepsilon^2} - \frac{1}{3}\frac{1}{\varepsilon}x\right)\sqrt{R}.$$

When the values found for $\varphi(0), \varphi(1)$ and $\varphi(2)$ are 0, the integral is algebraic; for instance, when $R = \frac{125}{256}\frac{\delta^4}{\varepsilon^3} + \frac{25}{32}\frac{\delta^3}{\varepsilon^2}x + \frac{15}{16}\frac{\delta^2}{\varepsilon}x^2 + \delta x^3 + \varepsilon x^4$, $\int\frac{x^4dx}{\sqrt{R}} = -\left(\frac{5}{12}\frac{\delta}{\varepsilon^2} - \frac{1}{3}\frac{1}{\varepsilon}x\right)\sqrt{R}$.

In a completely analogous manner, Abel reduces $\int \frac{dx}{(x-a)^m\sqrt{R}}$ to a linear combination of $\int\frac{dx}{\sqrt{R}}$, $\int\frac{xdx}{\sqrt{R}}$, $\int\frac{x^2dx}{\sqrt{R}}$, $\int\frac{dx}{(x-a)\sqrt{R}}$ and an algebraic function $Q\sqrt{R}$, Q having only one pole in a: $Q = \frac{\psi(1)}{x-a} + \frac{\psi(2)}{(x-a)^2} + \ldots + \frac{\psi(m-1)}{(x-a)^{m-1}}$. Indeed $d(Q\sqrt{R}) = S\frac{dx}{\sqrt{R}}$ with

$$S = \varphi(0) + \varphi(1)x + \varphi(2)x^2 + \frac{\chi(1)}{x-a} + \frac{\chi(2)}{(x-a)^2} + \ldots + \frac{\chi(m)}{(x-a)^m},$$
$$\varphi(0) = \left(\frac{1}{2}a\delta + \varepsilon a^2\right)\psi(1) - \frac{1}{2}(\delta + 4a\varepsilon)\psi(2) - \varepsilon\psi(3),$$
$$\varphi(1) = \frac{1}{2}\delta\psi(1), \quad \varphi(2) = \varepsilon\psi(1)$$

and $\chi(p) = -\alpha'(p-1)\psi(p-1) - \beta'(p-\frac{1}{2})\psi(p) - \gamma' p\psi(p+1) - \delta'(p+\frac{1}{2})\psi(p+2)$ $-\varepsilon'(p+1)\psi(p+3)$; here $\alpha' = \alpha + \beta a + \gamma a^2 + \delta a^3 + \varepsilon a^4$, $\beta' = \beta + 2\gamma a + 3\delta a^2 + 4\varepsilon a^3$, $\gamma' = \gamma + 3\delta a + 6\varepsilon a^2$, $\delta' = \delta + 4\varepsilon a$ and $\varepsilon' = \varepsilon$, so that $R = \alpha' + \beta'(x-a) + \gamma'(x-a)^2 + \delta'(x-a)^3 + \varepsilon'(x-a)^4$. In order to get the announced reduction, we determine the $\psi(p)$ by a linear system $\chi(p) = 0$ for $2 \le p \le m-1$, $\chi(m) = -1$; then $\varphi(0), \varphi(1), \varphi(2)$ are given by the preceding formulae and

$$\chi(1) = -\frac{1}{2}\beta'\psi(1) - \gamma'\psi(2) - \frac{3}{2}\delta'\psi(3) - 2\varepsilon'\psi(4).$$

For instance,

$$\int \frac{dx}{(x-a)^2\sqrt{fx}} = -\frac{\varepsilon a^2 + \frac{1}{2}\delta a}{fa}\int \frac{dx}{\sqrt{fx}} + \frac{\delta}{2fa}\int \frac{xdx}{\sqrt{fx}} + \frac{\varepsilon}{fa}\int \frac{x^2dx}{\sqrt{fx}} - \frac{1}{2}\frac{f'a}{fa}\int \frac{dx}{(x-a)\sqrt{fx}} - \frac{\sqrt{fx}}{(x-a)fa}, \tag{46}$$

where $R = fx$. This reduction does not work if $\alpha' = fa = 0$, which gives $\chi(m) = 0$ and in that case, we must take Q with a pole of order m in a and we see that $\int \frac{dx}{(x-a)^m\sqrt{R}}$ is reducible to $\int \frac{dx}{\sqrt{R}}, \int \frac{xdx}{\sqrt{R}}$ and $\int \frac{x^2dx}{\sqrt{R}}$ even for $m = 1$:

$$\int \frac{dx}{(x-a)\sqrt{R}} = -\frac{2\varepsilon a^2 + a\delta}{f'a}\int \frac{dx}{\sqrt{R}} + \frac{\delta}{f'a}\int \frac{xdx}{\sqrt{R}} + \frac{2\varepsilon}{f'a}\int \frac{x^2dx}{\sqrt{R}} - \frac{2}{f'a}\frac{\sqrt{R}}{x-a}. \tag{47}$$

The next task for Abel is to find the possible relations between integrals of the form

$$\int \frac{dx}{(x-b)\sqrt{R}}.$$

It is easy to see that the only possible relations have the form

$$\begin{aligned}&\varphi(0)\int \frac{dx}{(x-a)\sqrt{R}} + \varphi(1)\int \frac{dx}{(x-a')\sqrt{R}}\\ &+\varphi(2)\int \frac{dx}{(x-a'')\sqrt{R}} + \varphi(3)\int \frac{dx}{(x-a''')\sqrt{R}}\\ &= \sqrt{R}\left(\frac{A}{x-a} + \frac{A'}{x-a'} + \frac{A''}{x-a''} + \frac{A'''}{x-a'''}\right),\end{aligned}$$

where a, a', a'', a''' are the roots of R. Using the preceding reduction and the fact that $\int \frac{dx}{\sqrt{R}}, \int \frac{xdx}{\sqrt{R}}$ and $\int \frac{x^2dx}{\sqrt{R}}$ are independent, Abel finds $A = -\frac{2\varphi(0)}{f'a}$, $A' = -\frac{2\varphi(1)}{f'a'}$, $A'' = -\frac{2\varphi(2)}{f'a''}$, $A''' = -\frac{2\varphi(3)}{f'a'''}$, $A(2\varepsilon a^2 + a\delta) + A'(2\varepsilon a'^2 + a'\delta) + A''(2\varepsilon a''^2 + a''\delta) + A'''(2\varepsilon a'''^2 + a'''\delta) = 0$ and $A + A' + A'' + A''' = 0$; it is possible to choose $A''' = 0$ and this gives the relation

$$\begin{aligned}&\varphi(0)\int \frac{dx}{(x-a)\sqrt{R}} + \varphi(1)\int \frac{dx}{(x-a')\sqrt{R}} + \varphi(2)\int \frac{dx}{(x-a'')\sqrt{R}}\\ &= \sqrt{R}\left(\frac{A}{x-a} + \frac{A'}{x-a'} + \frac{A''}{x-a''}\right)\end{aligned}$$

with

$$\begin{aligned}\varphi(0) &= \frac{1}{2}(a-a')(a-a'')(a-a''')(a'-a'')(a'+a''-a-a'''),\\ \varphi(1) &= \frac{1}{2}(a'-a)(a'-a'')(a'-a''')(a''-a)(a+a''-a'-a'''),\\ \varphi(2) &= \frac{1}{2}(a''-a)(a''-a')(a''-a''')(a-a')(a+a'-a''-a''').\end{aligned}$$

Abel looks for linear relations between

$$\int \frac{dx}{\sqrt{R}}, \quad \int \frac{xdx}{\sqrt{R}}, \quad \int \frac{x^2dx}{\sqrt{R}} \quad \text{and} \quad \int \frac{dx}{(x-a)\sqrt{R}}, \quad \int \frac{dx}{(x-a')\sqrt{R}};$$

using (46), he finds $\int \frac{dx}{\sqrt{R}} = \frac{(a-a'')(a-a''')}{a''+a'''-a-a'} \int \frac{dx}{(x-a)\sqrt{R}} + \frac{(a'-a'')(a'-a''')}{a''+a'''-a-a'} \int \frac{dx}{(x-a')\sqrt{R}}$ and

$$\begin{aligned}
&\int \frac{x^2dx}{\sqrt{R}} + \frac{\delta}{2} \int \frac{xdx}{\sqrt{R}} \\
&= \frac{a'(a'-a-a''-a''')\cdot f'a}{2(a'-a)(a+a'-a''-a''')} \int \frac{dx}{(x-a)\sqrt{R}} \\
&\quad + \frac{a(a'-a-a''-a''')\cdot f'a'}{2(a-a')(a+a'-a''-a''')} \int \frac{dx}{(x-a')\sqrt{R}} \\
&\quad + \frac{\sqrt{R}}{(a-a')(a+a'-a''-a''')} \left(\frac{a'(a'-a-a''-a''')}{x-a} - \frac{a(a-a'-a''-a''')}{x-a'} \right).
\end{aligned}$$

When $a + a' = a'' + a'''$, these relations loose their sense and give $\int \frac{dx}{(x-a)\sqrt{R}} + \int \frac{dx}{(x-a')\sqrt{R}} = \frac{2\sqrt{R}}{(a''-a)(a''-a')(x-a)(x-a')}$.

In the second chapter of his memoir, Abel studies the integration of elliptic integrals by logarithmic functions. Such a function must be of the form

$$T = A\log(P + Q\sqrt{R}) + A'\log(P' + Q'\sqrt{R}) + \ldots + A^{(n)}\log(P^{(n)} + Q^{(n)}\sqrt{R})$$

with $P, Q, P', Q', \ldots$ polynomials and $A, A', \ldots$ constant, or, subtracting from dT a rational differential $T' = A\log\frac{P+Q\sqrt{R}}{P-Q\sqrt{R}} + A'\log\frac{P'+Q'\sqrt{R}}{P'-Q'\sqrt{R}} + \ldots$ Then dT' is a sum of terms of the form $\frac{M}{N}\cdot\frac{dx}{\sqrt{R}}$ with $M = A\frac{2N\frac{dP}{dx} - P\frac{dN}{dx}}{Q}$, $N = P^2 - Q^2R$; the fraction $\frac{M}{N}$ has only poles of order 1, these poles are not roots of R and it is easy to see that its polynomial part is of degree ≤ 1. Finally

$$T' = k\int \frac{dx}{\sqrt{R}} + k'\int \frac{xdx}{\sqrt{R}} + L\int \frac{dx}{(x-a)\sqrt{R}} + \ldots + L^{(\nu)}\int \frac{dx}{(x-a^{(\nu)})\sqrt{R}}, \tag{48}$$

and $\int \frac{x^2dx}{\sqrt{R}}$ cannot be reduced to other integrals by means of logarithms.

Let us suppose that T' contains $r+1$ logarithmic terms; looking at the degrees of $P, Q, P', Q', \ldots$ and at the corresponding number of indeterminate coefficients in T', Abel sees that the minimum value of ν is 2 and that r may be taken equal to 0. Moreover, one may take

$$P = f + f'x + f''x^2$$

of degree 2, $Q = 1$ and N of degree 2; then $f'' = \sqrt{\varepsilon}$, $f' = \frac{\delta}{2\sqrt{\varepsilon}}$ and $f = \frac{k(\delta^2 - 4\varepsilon\gamma) + 2k'\varepsilon\beta}{2(\delta k' - 4\varepsilon k)\sqrt{\varepsilon}}$. For $k = 0$ and $k' = 1$, one has

$$\int \frac{xdx}{\sqrt{R}} = (G + H\sqrt{K}) \int \frac{dx}{(x - \sqrt{K})\sqrt{R}} + (G - H\sqrt{K}) \int \frac{dx}{(x + \sqrt{K})\sqrt{R}}$$
$$+ \frac{1}{2\sqrt{\varepsilon}} \log \frac{\frac{\beta}{\delta}\sqrt{\varepsilon} + \frac{\delta}{2\sqrt{\varepsilon}}x + \sqrt{\varepsilon}x^2 + \sqrt{R}}{\frac{\beta}{\delta}\sqrt{\varepsilon} + \frac{\delta}{2\sqrt{\varepsilon}}x + \sqrt{\varepsilon}x^2 - \sqrt{R}},$$

where $G = -\frac{4\alpha\delta^2\varepsilon + \beta\delta^3 + 4\beta^2\varepsilon^2 - 4\beta\gamma\delta\varepsilon}{2(\delta^4 + 8\beta\delta\varepsilon^2 - 4\gamma\delta^2\varepsilon)}$, $H = -\frac{\delta}{4\varepsilon}$, $K = \frac{4\varepsilon}{\delta}\frac{\varepsilon\beta^2 - \alpha\delta^2}{4\gamma\delta\varepsilon - 8\beta\varepsilon^2 - \delta^3}$.

It is possible to have N of degree 1 when $(4\varepsilon\gamma - \delta^2)^2 + 4\delta^2(4\varepsilon\gamma - \delta^2) - 32\beta\delta\varepsilon^2 - 64\alpha\varepsilon^3 = 0$; then $f = \frac{4\varepsilon\gamma - \delta^2}{8\varepsilon\sqrt{\varepsilon}}$ and $\int \frac{xdx}{\sqrt{R}} = \frac{1}{3\varepsilon}(\mu' - \mu\delta) \int \frac{dx}{(x+\mu)\sqrt{R}}$ $+ \frac{1}{3\sqrt{\varepsilon}} \log \frac{\frac{\mu'}{\sqrt{\varepsilon}} + \frac{\delta}{2\sqrt{\varepsilon}}x + \sqrt{\varepsilon}x^2 + \sqrt{R}}{\frac{\mu'}{\sqrt{\varepsilon}} + \frac{\delta}{2\sqrt{\varepsilon}}x + \sqrt{\varepsilon}x^2 - \sqrt{R}}$, where $\mu' = \frac{4\varepsilon\gamma - \delta^2}{8\varepsilon}$ and $\mu = -\frac{\delta}{2\varepsilon}$.

Abel finds another reduction, writing $R = (p + qx + rx^2)(p' + q'x + x^2)$, $P = f(p' + q'x + x^2)$ and $Q = 1$ and choosing f such that

$$N = (f^2 - r)(p' + q'x + x^2)(x - a)^2;$$

then $\frac{M}{N} = 1 + L\frac{1}{x-a}$ with

$$L = \frac{pq' - qp' + (rq' - q)a^2}{(rq' - q)a}, \quad a = \frac{q - q'f^2}{2(f^2 - r)}.$$

This leads to the equation $f^4(q'^2 - 4p') - f^2(2qq' - 4p - 4p'r) + q^2 - 4pr = 0$ and to the relation $\int \frac{dx}{\sqrt{R}} = -L \int \frac{dx}{(x-a)\sqrt{R}} + A \log \frac{f(p'+q'x+x^2)+\sqrt{R}}{f(p'+q'x+x^2)-\sqrt{R}}$, where $A = \frac{f^2 - r}{f(rq' - q)}$.

Another formula is found by supposing $N = k(x - a)^4$; then

$$(p + p' - p'' - p''')a^2 - 2(pp' - p''p''')a + pp'(p'' + p''') - p''p'''(p + p') = 0,$$

where p, p', p'', p''' are the roots of R. In that case, $\frac{M}{N} = 1 - L\frac{1}{x-a}$, where

$$L = -\frac{2(f + af' + a^2 f'')}{f' + 2af''}, \quad f = \sqrt{pp'p''p''' + ka^4},$$
$$f' = -\frac{p + p' + p'' + p''' + 4ka}{2\sqrt{1 + k}}, \quad f'' = \sqrt{1 + k}$$

and $k = \frac{(p + p' - p'' - p''')^2}{(2(p'' + p''') - 4a)(2(p + p') - 4a)}$; so

$$\int \frac{dx}{\sqrt{(x - p)(x - p')(x - p'')(x - p''')}}$$
$$= L \int \frac{dx}{(x - a)\sqrt{(x - p)(x - p')(x - p'')(x - p''')}}$$
$$+ A \log \frac{f + f'x + f''x^2 + \sqrt{(x - p)(x - p')(x - p'')(x - p''')}}{f + f'x + f''x^2 - \sqrt{(x - p)(x - p')(x - p'')(x - p''')}},$$

with $A = \frac{1}{2\sqrt{(p+p'-2a)(p''+p'''-2a)}}$. All this work is inspired by Legendre's reduction of elliptic integrals to canonical forms as it is presented in the *Exercices de Calcul intégral* but Abel's study is deeper and more general for he investigates all the possible relations between such integrals and proves the independance of the three canonical kinds.

Abel also studies the general case, where $\frac{M}{N} = \frac{x^m + k^{(m-1)}x^{m-1} + \ldots + k}{x^m + l^{(m-1)}x^{m-1} + \ldots + l}$; if Q is of degree n, P must be of degree $n+2$ and $m \leq 2n+4$ which is the degree of N. With the notations $R = \varphi x$, $P = Fx$, $Q = fx, a, a', \ldots, a^{(m-1)}$ roots of N (with multiplicities $\mu, \mu', \ldots, \mu^{(m-1)}$), one has

$$Fa^{(j)} = \pm fa^{(j)}\sqrt{\varphi a^{(j)}} \quad (0 \leq j \leq m-1),$$

whence, by successive derivations, a linear system to determine the coefficients of P and Q. Then $x^m + k^{(m-1)}x^{m-1} + \ldots + k$ takes in $a^{(j)}$ the value $\pm A\sqrt{\varphi a^{(j)}} \cdot \psi a^{(j)}$, where

$$\psi x = (x-a)(x-a')\ldots(x-a^{(m-1)})\frac{dN}{Ndx};$$

this gives a linear system to get $k, k', \ldots$ in function of $A, a, a', \ldots$ For instance, when $\mu = \mu' = \ldots = \mu^{(m-1)} = 1$, $m = 2n+4 = 4$ if $Q = 1$ and Abel finds, for the coefficients of P,

$$\begin{aligned}
-f &= i\frac{a'a''a'''}{(a-a')(a-a'')(a-a''')}\sqrt{\varphi a} + i'\frac{aa''a'''}{(a'-a)(a'-a'')(a'-a''')}\sqrt{\varphi a'}\\
&\quad + i''\frac{aa'a'''}{(a''-a)(a''-a')(a''-a''')}\sqrt{\varphi a''} + i'''\frac{aa'a''}{(a'''-a)(a'''-a')(a'''-a'')}\sqrt{\varphi a'''},\\
f'' &= \frac{i\sqrt{\varphi a}}{(a-a')(a-a'')} + \frac{i'\sqrt{\varphi a'}}{(a'-a)(a'-a'')} + \frac{i''\sqrt{\varphi a''}}{(a''-a)(a''-a')},\\
f' &= \frac{i\sqrt{\varphi a}}{a-a'} + \frac{i'\sqrt{\varphi a'}}{a'-a} - (a+a')f'',
\end{aligned}$$

where i, i', i'', i''' are equal to ± 1, and $A = -\frac{1}{(a+a'+a''+a''')f''+2f'}$. When $m = 2$, $Q = 1$ and $P^2 - R = C(x-a)(x-a')^3$, he finds

$$\begin{aligned}
f'' &= \frac{1}{8}\frac{2\varphi a' \cdot \varphi'' a' - (\varphi' a')^2}{\varphi a'\sqrt{\varphi a'}},\\
f' &= \frac{1}{2}\frac{\varphi' a'}{\sqrt{\varphi a'}} - \frac{a'}{4}\frac{2\varphi a' \cdot \varphi'' a' - (\varphi' a')^2}{\varphi a'\sqrt{\varphi a'}},\\
f &= \sqrt{\varphi a} - \frac{a'}{2}\frac{\varphi' a'}{\sqrt{\varphi a'}} + \frac{a'^2}{8}\frac{2\varphi a' \cdot \varphi'' a' - (\varphi' a')^2}{\varphi a'\sqrt{\varphi a'}},\\
A &= -\frac{1}{(a+3a')f''+2f'}
\end{aligned}$$

and a, a' related by $\sqrt{\varphi a} \cdot \sqrt{\varphi a'} = \varphi a' + \frac{1}{2}(a-a')\varphi' a' + \frac{1}{8}(a-a')^2 \frac{2\varphi a' \cdot \varphi'' a' - (\varphi' a')^2}{\varphi a'}$. When $P^2 - R = C(x-a)^2(x-a')^2$,

$$f'' = \frac{1}{4}\frac{\varphi' a}{(a-a')\sqrt{\varphi a}} + \frac{1}{4}\frac{\varphi' a'}{(a'-a)\sqrt{\varphi a'}},$$
$$f' = \frac{1}{2}\frac{a'\varphi' a}{(a-a')\sqrt{\varphi a}} + \frac{1}{2}\frac{a\varphi' a'}{(a'-a)\sqrt{\varphi a'}},$$
$$f = \frac{1}{4}\frac{aa'}{a-a'}\frac{\varphi' a}{\sqrt{\varphi a}} + \frac{1}{4}\frac{aa'}{a'-a}\frac{\varphi' a'}{\sqrt{\varphi a'}} - \frac{a'\sqrt{\varphi a} - a\sqrt{\varphi a'}}{a-a'},$$
$$A = -\frac{2}{\frac{\varphi' a}{\sqrt{\varphi a}} + \frac{\varphi' a'}{\sqrt{\varphi a'}}}$$

and a, a' related by $(p+p'+p''+p''')aa' - (pp'-p''p''')(a+a') + pp'(p''+p''') - p''p'''(p+p') = 0$, where p, p', p'', p''' are the roots of R. So $\int \frac{dx}{\sqrt{\varphi x}} = -\int \frac{2b+2b'x}{(x-a)(x-a')}\frac{dx}{\sqrt{\varphi x}} + A\log\frac{P+\sqrt{\varphi x}}{P-\sqrt{\varphi x}}$, with $b = -2\frac{a'\sqrt{\varphi a}+a\sqrt{\varphi a'}}{\frac{\varphi' a}{\sqrt{\varphi a}}+\frac{\varphi' a'}{\sqrt{\varphi a'}}}$, $b' = 2\frac{\sqrt{\varphi a}+\sqrt{\varphi a'}}{\frac{\varphi' a}{\sqrt{\varphi a}}+\frac{\varphi' a'}{\sqrt{\varphi a'}}}$. In a third case $P^2 - R = C(x-p)(x-a)(x-a')^2$ and $P = (x-p)(f+f'x)$ and a' is function of a.

The last case considered by Abel is that in which $m = 1$. Here $P^2 - Q^2R = C(x-a)^{2n+4}$ and $\frac{M}{N} = \frac{x+k}{x-a}$ with $k = -a - \mu A\sqrt{\varphi a}$. The coefficients of P and Q are determined by a linear system and then a is given by an algebraic equation; this leads to

$$\int \frac{dx}{(x-a)\sqrt{R}} = \frac{1}{\mu A\sqrt{\varphi a}}\int \frac{dx}{\sqrt{R}} - \frac{1}{\mu\sqrt{\varphi a}}\log\frac{P+Q\sqrt{R}}{P-Q\sqrt{R}}.$$

Abel observes that the equation $P^2 - Q^2R = C(x-a)^{2n+4}$ is equivalent to $P'^2 - Q'^2R' = C$, where $F(x-a) = (x-a)^{n+2}P'\left(\frac{1}{x-a}\right)$, $f(x-a) = (x-a)^nQ'\left(\frac{1}{x-a}\right)$ and $\varphi(x-a) = (x-a)^4R'\left(\frac{1}{x-a}\right)$.

As we know, the same equation is met in the problem to express $\int \frac{(k+x)dx}{\sqrt{R}}$ by a logarithm $A\log\frac{P+Q\sqrt{R}}{P-Q\sqrt{R}}$; here $\frac{M}{N} = x+k$, so N is constant and may be taken as 1. The conditions of the problem are $x+k = 2A\frac{dP}{Qdx}$, $1 = P^2 - Q^2R$; the first method proposed by Abel to determine $P = f + f'x + \ldots + f^{(n+2)}x^{n+2}$ and $Q = e + e'x + \ldots + e^{(n)}x^n$ is that of indeterminate coefficients. The first condition gives $A = \frac{e^{(n)}}{(2n+4)f^{(n+2)}}$, $k = \frac{f'e^{(n)}}{(n+2)ef^{(n+2)}}$ and the second gives a system of $2n+5$ equations between the $2n+4$ coefficients $e^{(p)}$, $f^{(p)}$: $f^2 - \alpha e^2 = 1, \ldots, f^{(n+2)2} - \varepsilon e^{(n)2} = 0$. The compatibility of this system imposes a relation between the coefficients α, β, γ, δ and ε of R; for instance, when $n = 0$, so that $Q = e$ and $P = f + f'x + f''x^2$, one has

$$2ff' - \beta e^2 = f'^2 + 2ff'' - \gamma e^2 = 2f'f'' - \delta e^2 = 0,$$

whence $f'' = \frac{\delta\sqrt{\varepsilon}}{\sqrt{\beta^2\varepsilon - \alpha\delta^2}}$, $f' = \frac{\delta^2}{2\sqrt{\beta^2\varepsilon^2 - \alpha\varepsilon\delta^2}}$, $f = \frac{\beta\sqrt{\varepsilon}}{\sqrt{\beta^2\varepsilon - \alpha\delta^2}}$, $e = \frac{\delta}{\sqrt{\beta^2\varepsilon - \alpha\delta^2}}$ and $\gamma = \frac{\delta^2}{4\varepsilon} + \frac{2\beta\varepsilon}{\delta}$, $A = \frac{1}{4\sqrt{\varepsilon}}$, $k = \frac{\delta}{4\varepsilon}$.

But it is possible to get a *linear* system for the coefficients $e^{(p)}$, $f^{(p)}$: if

$$\begin{aligned} Fy &= fy^{n+2} + f'y^{n+1} + \ldots + f^{(n+2)}, \\ fy &= ey^n + e'y^{n-1} + \ldots + e^{(n)} \\ \text{and} \quad \varphi y &= \alpha y^4 + \beta y^3 + \gamma y^2 + \delta y + \varepsilon, \end{aligned}$$

the second condition is $(Fy)^2 - (fy)^2\varphi y = y^{2n+4}$ and it gives $Fy = fy \cdot \sqrt{\varphi y}$ when $y = 0$. The system is obtained by differentiating $2n + 3$ times this relation at $y = 0$. When $n = 0$, one finds $f'' = ce$, $f' = c'e$, $f = \frac{c''}{2}e, 0 = c'''$, where $c^{(p)} = \frac{d^p\sqrt{\varphi y}}{dy^p}\big|_{y=0}$ and $\gamma = \frac{\delta^2}{4\varepsilon} + \frac{2\beta\varepsilon}{\delta}$ as above; when $n = 1$, the system is $0 = 2c' + c''\frac{e'}{e}$, $0 = 4c''' + c''''\frac{e'}{e}$, $0 = 5c'''' + c'''''\frac{e'}{e}$, whence

$$c'c'''' - 2c''c''' = 2c'c''''' - 5c''c'''' = 0.$$

Without restricting the generality, we may take $\varepsilon = 1$ and $\beta = -\alpha$; the preceding equations then give $\delta = 2$, $\gamma = -3$ and finally

$$\begin{aligned} &\int \frac{xdx}{\sqrt{x^4 + 2x^3 + 3x^2 - \alpha x + \alpha}} \\ &= \frac{1}{6} \log \frac{x^3 + 3x^2 - 2 - \frac{\alpha}{2} + (x+2)\sqrt{x^4 + 2x^3 - 3x^2 - \alpha x + \alpha}}{x^3 + 3x^2 - 2 - \frac{\alpha}{2} - (x+2)\sqrt{x^4 + 2x^3 - 3x^2 - \alpha x + \alpha}}. \end{aligned} \tag{49}$$

Abel proposes another way to study the equation $P^2 - Q^2R = 1$; he writes it $P + 1 = P'^2R'$, $P - 1 = Q'^2R''$, where $P'Q' = Q$ and $R'R'' = R$. Then $P = \frac{1}{2}(P'^2R' + Q'^2R'')$ and $2 = P'^2R' - Q'^2R''$; with $R' = x^2 + 2qx + p$, $R'' = x^2 + 2q'x + p'$ and P', Q' constant, one finds $q = q'$, $P' = Q' = \frac{\sqrt{2}}{\sqrt{p-p'}}$, $P = \frac{2x^2+4qx+p+p'}{p-p'}$, $Q = \frac{2}{p-p'}$, $k = q$ and $A = \frac{1}{4}$, so

$$\begin{aligned} &\int \frac{(x+q)dx}{\sqrt{(x^2 + 2qx + p)(x^2 + 2qx + p')}} \\ &= \frac{1}{4} \log \frac{2x^2 + 4qx + p + p' + 2\sqrt{R}}{2x^2 + 4qx + p + p' - 2\sqrt{R}}. \end{aligned}$$

With $P' = \frac{x+m}{c}$, $Q' = \frac{x+m'}{c}$, one finds $2q = r + m' - m$, $2q' = r + m - m'$,

$$\begin{aligned} p &= \frac{1}{2}r(3m' - m) + \frac{1}{2}m^2 - \frac{1}{2}m'^2 - mm', \\ p' &= \frac{1}{2}r(3m - m') + \frac{1}{2}m'^2 - \frac{1}{2}m^2 - mm', \\ 2c^2 &= \frac{1}{2}r(m' - m)^3 + \frac{1}{2}(m - m')(m^3 - m^2m' - m'^2m + m'^3), \end{aligned}$$

where $r = q + q'$, and then

$$P = \frac{(x^2+2mx+m^2)(x^2+2qx+p)-c^2}{c^2},$$
$$Q = \frac{x^2+(m+m')x+mm'}{c^2},$$
$$k = \frac{1}{4}(3r-m'-m).$$

If we impose $k=0$, $r=\frac{m+m'}{3}$, $m=2q'+q$, $m'=2q+q'$, $p=-q^2-2qq'$ and $p'=-q'^2-2qq'$; we have

$$\int \frac{xdx}{\sqrt{(x^2+2qx-q^2-2qq')(x^2+2q'x-q'^2-2qq')}}$$
$$= \frac{1}{4}\log\frac{(x+q+2q')\sqrt{x^2+2qx-q^2-2qq'}+(x+q'+2q)\sqrt{x^2+2q'x-q'^2-2qq'}}{(x+q+2q')\sqrt{x^2+2qx-q^2-2qq'}-(x+q'+2q)\sqrt{x^2+2q'x-q'^2-2qq'}}.$$

The second method to study the equation $P^2-Q^2R=1$ is that used by Abel in his published memoir for the more general case of hyperelliptic integrals: putting $R=r^2+s$, with r of degree 2 and s of degree 1, the equation becomes $P^2-Q^2r^2-Q^2s=1$ and it shows that $P=Qr+Q_1$ with $\deg Q_1<\deg P$. Then $Q_1^2+2QQ_1r-Q^2s=1$ or, if $r=sv+u$, with v of degree 1 and u constant, $Q_1^2+2QQ_1u+Qs(2vQ_1-Q)=1$; thus $Q_2=Q-2vQ_1$ if of degree $<\deg Q=n$ and

$$s_1Q_1^2-2r_1Q_1Q_2-sQ_2^2=1,$$

with $s_1=1+4uv$, $r_1=r-2u$, $\deg Q_1=n-1$ and $\deg Q_2=n-2$. Iterating the process, one gets equations $s_{2\alpha-1}Q_{2\alpha+1}^2-2r_{2\alpha}Q_{2\alpha}Q_{2\alpha+1}-s_{2\alpha}Q_{2\alpha}^2=1$, $s_{2\alpha'+1}Q_{2\alpha'+1}^2-2r_{2\alpha'+1}Q_{2\alpha'+1}Q_{2\alpha'+2}-s_{2\alpha'}Q_{2\alpha'+2}^2=1$, with $\deg Q_p=n-p$; this gives $s_nQ_n^2=(-1)^{n+1}$, Q_n and s_n constant. The induction relations to determine the s_m are

$$s_m=s_{m-2}+4u_{m-1}v_{m-1},\ r_m=r_{m-1}-2u_{m-1}=s_mv_m+u_m. \tag{50}$$

A consequence of these relations is that $s_{m-1}s_m+r_m^2=s_{m-1}s_{m-2}+r_{m-1}^2$, so that this quantity does not depend on m and

$$s_{m-1}s_m+r_m^2=ss_1+r_1^2=r^2+s=R; \tag{51}$$

as $s_n=\mu$ is constant, it is easy to see that $r_{n-k}=r_k$, $s_{n-k}=s_{k-1}\mu^{(-1)^k}$, $v_{n-k}=v_{k-1}\mu^{(-1)^{k-1}}$ and $u_{n-k}=-u_{k-1}$. For $n=2\alpha+1$ and $k=\alpha+1$, this gives $\mu=1$ and $u_\alpha=0$; for $n=2\alpha$, $u_{\alpha-1}+u_\alpha=0$. The Q_m are determined from Q_n by the induction relations $Q_m=2v_mQ_{m+1}+Q_{m+2}$ and we see that $r, 2v, 2v_1, \ldots, 2v_{n-1}$ are the partial quotients of the continued fraction for $\frac{P}{Q}$, which is obtained by truncating that for $\sqrt{R}$. Putting $r_m=x^2+ax+b_m$, $s_m=c_m+p_mx$, $v_m=(g_m+x)\frac{1}{p_m}$ and $q_m=b-b_m$, Abel draws from (50) and (51) the relations

$$q_m = \frac{\frac{1}{2}p^2 + (ap - 2c)q_{m-1} - q_{m-2}q_{m-1}^2}{q_{m-1}^2}, \quad \frac{c_{m-1}}{p_{m-1}} = \frac{c + q_{m-1}q_m}{p}$$

and $\quad p_m p_{m-1} = 2q_m$;

since $u_m = \frac{b_m - b_{m+1}}{2} = \frac{1}{2}(q_{m+1} - q_m)$ and $g_m = a - \frac{c_m}{p_m}$, these relations allow to determine r_m, s_m, u_m and v_m if we know the q_m, which are determined by an induction relation starting from $q = 0, q_1 = 2\frac{bp^2 - acp + c^2}{p^2}$ and are rational functions of a, b, c, p.

Abel applies this method to the elliptic integral $\int \frac{(x+k)dx}{\sqrt{(x^2+ax+p)^2+px+c}}$. The condition $s_n =$ constant is equivalent to $p_n = 0$ and it leads to $q_n = 0$ and $q_{n-k} = q_k$. The coefficient k is equal to $\frac{1}{n+2}a + \frac{1}{n+2}\left(\frac{c}{p} + \frac{c_1}{p_1} + \ldots + \frac{c_{n-1}}{p_{n-1}}\right)$ and the polynomials P and Q are determined by the continued fraction. When $c = 0$, Abel finds the results published in his 1826 paper, using

$$q_1 = 2b, \quad q_2 = \frac{p(p + 4ab)}{8b^2},$$

$$q_3 = \frac{2b(16b^3 - p(p + 4ab))}{(p + 4ab)^2},$$

$$q_4 = \frac{4bp(p + 4ab)(p^2 + 6abp + 8a^2b^2 - 8b^3)}{(16b^3 - p(p + 4ab))^2}.$$

From a relation $\int \frac{(y+k')dy}{\sqrt{R'}} = A' \log \frac{P'+Q'\sqrt{R'}}{P'-Q'\sqrt{R'}}$, Abel deduces

$$\int \frac{x + k}{x + l}\frac{dx}{\sqrt{R}} = A \log \frac{P + Q\sqrt{R}}{P - Q\sqrt{R}}$$

through the change of variable $y = \frac{1}{x+l}$; he finds $k = l + \frac{1}{k'}$, $A = -\frac{A'}{k'}$ and an algebraic equation to determine l in function of the coefficients of R. Indeed, when

$$R' = (y^2 + ay + b)^2 + c + py \quad \text{and} \quad R = (b^2 + c)(x^4 + \delta x^3 + \gamma x^2 + \beta x + \alpha),$$

$$2ab + p = (b^2 + c)(\delta - 4l), a^2 + 2b = (b^2 + c)(\gamma - 3\delta l + 6l^2),$$

$$2a = (b^2 + c)(\beta - 2\gamma l + 3\delta l^2 - 4l^3)$$

and $1 = (b^2+c)(\alpha - \beta l + \gamma l^2 - \delta l^3 + l^4)$. From this Abel deduces, with $-l$ instead of l, $\int \frac{dx}{(x-l)\sqrt{R}} = -\frac{1}{l+k}\int \frac{dx}{\sqrt{R}} - \frac{1}{(2n+4)\sqrt{\alpha+\beta l+\gamma l^2+\delta l^3+l^4}} \log \frac{P+Q\sqrt{R}}{P-Q\sqrt{R}}$, which gives a new proof of (49) when $l + k = \infty$.

In the third chapter of the *Théorie des transcendantes elliptiques*, Abel shows that the periods of an integral of the third kind $p = \int \frac{dx}{(x-a)\sqrt{R}}$ are combinations of the periods of the integrals $\int \frac{dx}{\sqrt{R}}$, $\int \frac{xdx}{\sqrt{R}}$ and $\int \frac{x^2dx}{\sqrt{R}}$. Taking the integral from a value $x = r$ which annihilates $R = fx$, differentiating with respect to a and using (46), he obtains

$$\frac{dp}{da}+\frac{1}{2}\frac{f'a}{fa}p=\frac{\sqrt{fx}}{(a-x)fa}+\frac{1}{fa}\int\frac{dx}{\sqrt{fx}}(A+Bx+Cx^2)$$

where $A=-\varepsilon a^2-\frac{1}{2}\delta a$, $B=\frac{1}{2}\delta$ and $C=\varepsilon$. From this he deduces

$$p\sqrt{fa}-\sqrt{fx}\int\frac{da}{(a-x)\sqrt{fa}}=\int\frac{da}{\sqrt{fa}}\int\frac{dx}{\sqrt{fx}}(A+Bx+Cx^2)+\text{constant}$$

and the constant is seen to be 0 by making $a=r$. Thus

$$\begin{aligned}&\sqrt{fa}\int\frac{dx}{(x-a)\sqrt{fx}}-\sqrt{fx}\int\frac{da}{(a-x)\sqrt{fa}}\\&=\int\frac{da}{\sqrt{fa}}\int\frac{(\frac{1}{2}\delta x+\varepsilon x^2)dx}{\sqrt{fx}}-\int\frac{dx}{\sqrt{fx}}\int\frac{(\frac{1}{2}\delta a+\varepsilon a^2)da}{\sqrt{fa}}\end{aligned}$$

which the formula (42) for the case of elliptic integrals. When r' is another root of fx, one obtains $\sqrt{fa}\int_r^{r'}\frac{dx}{(x-a)\sqrt{fx}}=\int_r\frac{da}{\sqrt{fa}}\int_r^{r'}\frac{(\frac{1}{2}\delta x+\varepsilon x^2)dx}{\sqrt{fx}}-\int_r^{r'}\frac{dx}{\sqrt{fx}}\int_r\frac{(\frac{1}{2}\delta a+\varepsilon a^2)da}{\sqrt{fa}}$. And if r'' is a third root of fx, $\int_r^{r''}\frac{da}{\sqrt{fa}}\int_r^{r'}\frac{(\frac{1}{2}\delta x+\varepsilon x^2)dx}{\sqrt{fx}}=\int_r^{r'}\frac{dx}{\sqrt{fx}}\int_r^{r''}\frac{(\frac{1}{2}\delta a+\varepsilon a^2)da}{\sqrt{fa}}$.

Abel finds new relations between periods starting from

$$\begin{aligned}s&=A\log\frac{P+Q\sqrt{R}}{P-Q\sqrt{R}}+A'\log\frac{P'+Q'\sqrt{R}}{P'-Q'\sqrt{R}}+\dots\\&=\int\frac{B+Cx}{\sqrt{R}}dx+L\int\frac{dx}{(x-a)\sqrt{R}}+L'\int\frac{dx}{(x-a')\sqrt{R}}+\dots\end{aligned}$$

(cf. (48)) which gives, by integrating from r to r':

$$\begin{aligned}s'-s=&\int_r^{r'}\frac{B+Cx}{\sqrt{fx}}dx\\&-\int_r^{r'}\frac{dx}{\sqrt{fx}}\left(\frac{L}{\sqrt{fa}}\int_r\frac{(\frac{1}{2}\delta a+\varepsilon a^2)da}{\sqrt{fa}}+\frac{L'}{\sqrt{fa'}}\int_r\frac{(\frac{1}{2}\delta a'+\varepsilon a'^2)da'}{\sqrt{fa'}}+\dots\right)\\&+\int_r^{r'}\frac{(\frac{1}{2}\delta x+\varepsilon x^2)dx}{\sqrt{fx}}\left(\frac{L}{\sqrt{fa}}\int_r\frac{da}{\sqrt{fa}}+\frac{L'}{\sqrt{fa'}}\int_r\frac{da'}{\sqrt{fa'}}+\dots\right)\end{aligned}$$

The end of the *Théorie des transcencantes elliptiques* (p. 173–188) is devoted to the proof that an integral of the third kind $\Pi(n)=\int\frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-c^2x^2)}}$ may be transformed in a linear combination of the integral of the first kind $F=\int\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$, some logarithms (or arctangents) of algebraic functions and another integral of the third kind $\Pi(n')$ with a parameter n' arbitrarily large or arbitrarily close to a certain limit, and more generally to the relations between integrals of the third kind with different parameters. Let us consider $s = \arctan \frac{\sqrt{R}}{Q}$, with $R = (1-x^2)(1-c^2x^2)$ and $Q = x(a+bx^2)$; we have $ds = \frac{M}{N}\frac{dx}{\sqrt{R}}$ with $N = Q^2 + R$ and $M = \frac{1}{2}Q\frac{dR}{dx} - R\frac{dQ}{dx}$. If we impose that $N = k(1+nx^2)(1+n_1x^2)^2$, we find that $k = 1, b = \pm n_1\sqrt{n}, a = (1+n_1)\sqrt{1+n} \mp n_1\sqrt{n} = \chi(n)$ and $n_1 = \pm(\sqrt{1+n} \pm \sqrt{n})(\sqrt{c^2+n} \pm \sqrt{n}) = f(n)$. Then $\frac{M}{N} = A + \frac{L}{1+nx^2} + \frac{L'}{1+n_1x^2}$ with $A = 2a - \left(\frac{1}{n} + \frac{2}{n_1}\right)b$, $L = \frac{n_1}{\sqrt{n}} - a$ and $L' = 2\sqrt{n} - 2a$. Thus

$$\begin{aligned}\Pi(n) &= \frac{\sqrt{n}}{n_1 - a\sqrt{n}} \arctan \frac{\sqrt{R}}{ax+bx^3} - \frac{2a\sqrt{n} - (2n+n_1)}{n_1 a\sqrt{n}} F \\ &\quad + \frac{(2a - 2\sqrt{n})\sqrt{n}}{n_1 - a\sqrt{n}} \Pi(n_1) + C \\ &= \beta F + \gamma \Pi(n_1) + \alpha \arctan \frac{ax+bx^3}{\sqrt{R}}\end{aligned} \tag{52}$$

with $\alpha = \frac{\pm\sqrt{n}}{n_1 \mp a\sqrt{n}} = \varphi(n)$, $\beta = -\frac{\pm 2a\sqrt{n} - 2n - n_1}{n_1 \mp a\sqrt{n}} = \theta(n)$, $\gamma = \frac{\pm(2a \mp 2\sqrt{n})\sqrt{n}}{n_1 \mp a\sqrt{n}} = \psi(n)$. It is easy to see that $n_1 > 4n$ and that χ is an increasing function when both upper signs are chosen in $f(n)$. Thus, iterating the operation, we arrive at a parameter n_m as large as we wish, with α_m equivalent to $\frac{1}{\sqrt{n_m}}$, β_m remaining between 0 and 1 and $\lim \beta_m = 0$, $\lim \gamma_m = 4$. On the contrary, when both lower signs are chosen, n_m decreases and its limit is the root k of the equation $k = (\sqrt{k+1} - \sqrt{k})(\sqrt{k+c^2} - \sqrt{k})$. Applying (52) to $\Pi(k)$, we obtain

$$\Pi(k) = \frac{2a + 3\sqrt{k}}{3(a+\sqrt{k})} F - \frac{1}{3(a+\sqrt{k})} \arctan \frac{ax - k^{\frac{3}{2}} x^3}{\sqrt{R}}.$$

The formulae $n_1 = -(\sqrt{1+n} + \sqrt{n})(\sqrt{c^2+n} - \sqrt{n})$ and

$$n_1 = -(\sqrt{1+n} - \sqrt{n})(\sqrt{c^2+n} + \sqrt{n})$$

respectively lead to values of n_m between $-c^2$ and $-c$ and between -1 and $-c$. Abel also studies the case in which n is negative.

The transformed parameter n_1 is given by an equation of degree 4; inversely, one has

$$n = \frac{(n_1^2 - c^2)^2}{4n_1(n_1+1)(n_1+c^2)}.$$

When the sequence (n_m) is periodic, the integrals $\Pi(n_m)$ may be expressed as combinations of F and some arctangents.

Abel finds other relations as

$$\Pi(n) = -\frac{m'}{\mu}\frac{\psi(n_1)}{\psi(n)}\Pi(n_1) - \frac{A}{\mu n n_1 \psi(n)} F + \frac{1}{\mu\psi(n)} \arctan \frac{Q\sqrt{R}}{P},$$

with P, Q polynomials such that

$$P^2 + Q^2 R = (1 + nx^2)^\mu (1 + n_1 x^2)^{\mu'}, \quad \psi(n) = \frac{\sqrt{(1+n)(c^2+n)}}{\sqrt{n}},$$

A constant and $n_1 = \chi(n)$ a certain function. When for instance $P = 1 + bx^2$ and $Q = ex$, $\chi(n) = \frac{c(c-\sqrt{c^2+n})(1-\sqrt{1+n})}{n}$.

An essential discovery of Abel in the theory of elliptic functions is that these functions, obtained by inverting elliptic integrals, have 2 independent periods in the complex domain. In his posthumous memoir *Propriétés remarquables de la fonction* $y = \varphi x$ *déterminée par l'équation* $fydy - dx\sqrt{(a-y)(a_1-y)(a_2-y)\dots(a_m-y)} = 0$*, fy étant une fonction quelconque de y qui ne devient pas nulle ou infinie lorsque* $y = a, a_1, a_2, \dots, a_m$ (*Œuvres*, t. II, p. 40–42), he shows that the function φx, which is the inverse function of the hyperelliptic integral $x = \int \frac{fydy}{\sqrt{\psi y}}$ where $\psi y = (a-y)(a_1-y)(a_2-y)\dots(a_m-y)$, must have each of the numbers $2(\alpha - \alpha_k)$ as period, where α_k is the values of the integral corresponding to $y = a_k$. Jacobi later proved (1834) that a regular uniform function of one complex variable cannot have more than 2 independent periods; thus the inverse function of a hyperelliptic integral cannot be uniform when $m > 4$. The inversion problem for hyperelliptic integrals or more generally for abelian integrals must involve functions of several complex variable, as Jacobi (1832) discovered through his intertretation of Abel theorem. Here Abel writes the Taylor series for the function φ:

$$\varphi(x+v) = y + v^2 Q_2 + v^4 Q_4 + v_6 Q_6 + \dots + \sqrt{\psi y}(vQ_1 + v^3 Q_3 + v^5 q_5 + \dots)$$

where the Q_j do not have poles at the a_k. Thus $\varphi(\alpha + v) = a + v^2 Q_2 + v^4 Q_4 + v_6 Q_6 + \dots \varphi(\alpha + v)$ is an even function of v and $\varphi(2\alpha - v) = \varphi v$. In the same way $\varphi(2\alpha_1 - v) = \varphi v$ and $\varphi(2\alpha - 2\alpha_1 + v) = \varphi v$ and so on.

5 Abel Theorem

The most famous of Abel's results is a remarkable extension of Euler addition theorem for elliptic integrals. It is known as *Abel theorem* and gives the corresponding property for any integral of an algebraic function; such integrals are now called *abelian integrals*. This theorem, sent to the french Academy of Sciences by Abel in 1826 in a long memoir titled *Mémoire sur une propriété générale d'une classe très-étendue de fonctions transcendantes*, is rightly considered as the base of the following developments in algebraic geometry. Due to the negligence of the french Academicians, this fundamental memoir was published only in 1841, after the first edition of Abel's Work (1839).

In the introduction, Abel gives the following statement:

"When several functions are given of which the derivatives may be roots of the *same algebraic equation*, of which all the coefficients are *rational* functions of the same variable, one can always express the sum of any number of such functions by an *algebraic* and *logarithmic* function, provided that a certain number of *algebraic* relations be prescribed between the variables of the functions in question."

He adds that the number of relations does not depend on the number of the functions, but only on their nature. It is 1 for the elliptic integrals, 2 for the functions of which the derivatives contains only the square root of a polynmial of degree ≤ 6 as irrationality.

A second statement, which is properly Abel theorem says:

"One may always express the sum of a *given* number of functions, each of which is multiplied by a rational number, and of which the variable are arbitrary, by a similar sum of a *determined* number of functions, of which the variables are algebraic functions of the variables of the givent functions."

The proof of the first statement is short (§ 1–3, p. 146–150). Abel considers an algebraic equation $0 = p_0 + p_1 y + p_2 y^2 + \ldots + p_{n-1} y^{n-1} + y_n = \chi y$ with coefficients polynomials in x; this equation is supposed to be irreducible. He introduces another polynomial

$$\theta y = q_0 + q_1 y + q_2 y^2 + \ldots + q_{n-1} y^{n-1}$$

in x, y, certain coefficients $a, a', a'', \ldots$ of the polynomials $q_0, q_1, \ldots, q^{n-1}$ being indeterminates. The resultant $r = \theta y' \theta y'' \ldots \theta y^{(n)}$ of χ and θ, where $y', y'', \ldots, y^{(n)}$ are the roots of $\chi y = 0$, is a polynomial in $x, a, a', a'', \ldots$, which may be decomposed in $r = F_0 x F x$ where $F_0 x$ and Fx are polynomials in x and $F_0 x$ does not depend on $a, a', a'', \ldots$. Let $x_1, x_2, \ldots, x_\mu$ be the roots of $Fx = 0$ and $y_1, y_2, \ldots, y_\mu$ the corresponding common roots of the equations $\chi y = 0$, $\theta y = 0$. The y_k are rational functions of $x_k, a, a', a'', \ldots$ by the theory of elimination. Now let $f(x, y)dx$ be a differential form, with f a rational function of x, y. When $Fx = 0$, $dx = -\frac{\delta Fx}{F'x}$ where $F'x$ is the derivative with respect to x and δFx the differential with respect to $a, a', a'', \ldots$ Thus $f(x_k, y_k)dx_k = -\frac{f(x_k, y_k)}{F' x_k} \delta F x_k$ is rational with respect to $x_k, a, a', a'', \ldots$ and $dv = f(x_1, y_1)dx_1 + f(x_2, y_2)dx_2 + \ldots + f(x_\mu, y_\mu)dx_\mu$ is rational with respect to $a, a', a'', \ldots$ A consequence is that the function

$$\int f(x_1, y_1)dx_1 + \int f(x_2, y_2)dx_2 + \ldots + \int f(x_\mu, y_\mu)dx_\mu = v \tag{53}$$

is an algebraic and logarithmic function of $a, a', a'', \ldots$ Now if there are α indeterminate coefficients $a, a', a'', \ldots$ in θy, they may be determined by arbitrarily choosing α couples (x_j, y_j) of roots of $\chi y = 0$ and writing the equations $\theta y_j = 0$; the other y_k are then rational functions of x_k and the (x_j, y_j).

Abel gives a cleaver way to do the computation (§ 4, p. 150–159), first writing $\delta F x = \frac{\delta r}{F_0 x} = \frac{r \delta \theta y}{F_0 x \theta y}$ and

$$\begin{aligned} &f(x, y)dx \\ &= -\frac{1}{F_0 x F' x}\left(f(x, y')\frac{r}{\theta y'}\delta\theta y' + f(x, y'')\frac{r}{\theta y''}\delta\theta y'' + \ldots + f(x, y^{(n)})\frac{r}{\theta y^{(n)}}\delta\theta y^{(n)} \right) \\ &= -\frac{1}{F_0 x F' x} \sum f(x, y)\frac{r}{\theta y}\delta\theta y \end{aligned}$$

where the sum is extended to the n roots $y', y'', \ldots, y^{(n)}$. Then he writes $f(x, y) = \frac{f_1(x,y)}{f_2x\chi'y}$ where $f_1(x, y)$ is a polynomial in x, y, of degree $\leq n-1$ in y, f_2x a polynomial in x and $\chi'y$ the derivative of χy with respect to y. If $f_1(x, y)\frac{r}{\theta y}\delta\theta y = R^{(1)}y + Rx \cdot y^{n-1}$, with $R^{(1)}y$ polynomial in x, y of degree $\leq n-2$ in y and Rx a polynomial in x, it is easy to see that

$$\sum \frac{f_1(x, y)}{\chi'y}\frac{r}{\theta y}\delta\theta y = Rx. \tag{54}$$

Thus $dv = \sum f(x, y)dx = -\sum \frac{R_1x}{f_2x\cdot F_0x\cdot F'x}$; here the sum is extended to the μ roots x_k of $Fx = 0$. Grouping the roots of F_0x and of f_2x, we obtain $dv = -\sum \frac{R_1x}{\theta_1x\cdot F'x}$ where R_1x has no common root either with F_0x or with f_2x and the roots of θ_1x annihilate F_0x or f_2x. If R_2x is the quotient and R_3x the remainder of the Euclidian division of R_1x by θ_1x, one computes that $\sum \frac{R_1x}{F'x}$ is the coefficient of $\frac{1}{x}$ in the expansion of $\frac{R_1x}{\theta_1x\cdot Fx}$ in decreasing powers of x; this result comes from the development of $\frac{1}{F\alpha} = \sum \frac{1}{\alpha-x}\frac{1}{F'x}$ in decreasing powers of α. A rather more complicated computation, based on the decomposition of $\frac{R_3x}{\theta_1x}$ in simple elements, gives $\sum \frac{R_3x}{\theta_1x\cdot F'x} = -\sum{}' \nu\frac{d^{\nu-1}}{d\beta^{\nu-1}}\left(\frac{R_1\beta}{\theta_1^{(\nu)}\beta\cdot F\beta}\right)$ where the sum of the right hand side is extended to the roots β of θ_1x and, for each β, ν is the multiplicity of β . Unfortunately this result is incorrect and Sylow corrects it in the notes at the end of the second volume of Abel's Works (p. 295–296). The correct result is

$$\sum \frac{R_3x}{\theta_1x\cdot F'x} = \sum{}' \frac{1}{\Gamma\nu}\frac{d^{\nu-1}}{d\beta^{\nu-1}}\left(\frac{R_1\beta}{\vartheta\beta}\sum\frac{1}{(x-\beta)Fx}\right)$$

where $\vartheta x = \frac{\theta_1x}{(x-\beta)^\nu}$. From (54), we draw $R_1x = F_2x\cdot Fx\sum\frac{f_1(x,y)}{\chi'y}\frac{\delta\theta y}{\theta y}$ where $F_2x = \frac{\theta_1x}{f_2x}$ is a rational function of x independent of $a, a', a'', \ldots$ as are θ_1x, $f_1(x, y)$ and $\chi'y$. Thus $dv = -\prod\frac{F_2x}{\theta_1x}\sum\frac{f_1(x,y)}{\chi'y}\frac{\delta\theta y}{\theta y} + \sum{}'\frac{1}{\Gamma\nu}\frac{d^{\nu-1}}{dx^{\nu-1}}\left(\frac{F_2x}{\vartheta x}\sum\frac{f_1(x,y)}{\chi'y}\frac{\delta\theta y}{\theta y}\right)$ where the symbol $\prod$ denotes the coefficient of $\frac{1}{x}$ in the expansion of the following function in decreasing powers of x. Now the expression in the right hand side is integrable and gives

$$v = C - \prod\frac{F_2x}{\theta_1x}\sum\frac{f_1(x, y)}{\chi'y}\log\theta y + \sum{}'\frac{1}{\Gamma\nu}\frac{d^{\nu-1}}{dx^{\nu-1}}\left(\frac{F_2x}{\vartheta x}\sum\frac{f_1(x, y)}{\chi'y}\log\theta y\right). \tag{55}$$

In general $F_0x = 1$ and then $F_2x = 1, \theta_1x = f_2x$. If for example $f_2x = (x-\beta)^m$, the formula (55) takes the form

$$\sum\int\frac{f_1(x, y)dx}{(x-\beta)^m\chi'y} = C - \prod\sum\frac{f_1(x, y)}{(x-\beta)^m\chi'y}\log\theta y + \frac{1}{1\cdot 2\cdots(m-1)}\frac{d^{m-1}}{d\beta^{m-1}}\left(\sum\frac{f_1(\beta, B)}{\chi'B}\log\theta B\right)$$

where B is the value of y when $x = \beta$ (the second term disappears if $m = 0$).

In the fifth paragraph (p. 159–170), Abel studies under which conditions the right hand side of (54) is a constant independent of $a, a', a'', \ldots$ He supposes that $F_0 x = 1$; then $\theta_1 x = f_2 x$ must be constant and $\sum \int \frac{f_1(x,y)dx}{(x-\beta)^m \chi' y} = C - \prod \sum \frac{f_1(x,y)}{\chi' y} \log \theta y$. In order that this expression be constant, he finds that the following condition must be realised: $\sup_{1 \leqslant k \leqslant n} h \frac{f_1(x, y^{(k)})}{\chi' y^{(k)}} < -1$ where, for any function R of x, hR denotes the highest exponent of x in the expansion of R in decreasing powers of x. This condition is equivalent to $hf_1(x, y^{(k)}) < h\chi' y^{(k)} - 1$ for $1 \le k \le n$ and Abel deduces from it that $h(t_m y^{(k)} m) < h\chi' y^{(k)} - 1$ for $0 \le m \le n-1$ and $1 \le k \le n$ if

$$f_1(x, y) = t_0 + t_1 y + t_2 y^2 + \ldots + t_{n-1} y^{n-1}.$$

A proof of this deduction is given by Sylow in the notes (*Œuvres*, t. II, p. 296–297). The condition now takes the form $ht_m < \inf_{1 \leqslant k \leqslant n} (h\chi' y^{(k)} - mhy^{(k)}) - 1$ for $0 \le m \le n-1$ and $1 \le k \le n$. Abel arranges the $y^{(k)}$ in a way such that $hy' \le hy'' \le \ldots \le hy^{(n)}$. Thus, in general $h(y^{(k)} - y^{(\ell)}) = hy^{(k)}$ for $\ell > k$ and $h\chi' y^{(k)} = hy' + hy'' + \ldots + hy^{(k-1)} + (n-k)hy^{(k)}$. Now one sees that

$$\inf_{1 \leqslant k \leqslant n} (h\chi' y^{(k)} - mhy^{(k)}) = hy' + hy'' + \ldots + hy^{(n-m-1)},$$

so that $ht_m = hy' + hy'' + \ldots + hy^{(n-m-1)} - 2 + \varepsilon_{n-m-1}$ with $0 \le \varepsilon_{n-m-1} < 1$. Let us suppose that

$$hy^{(j)} = \frac{m^{(\alpha)}}{\mu^{(\alpha)}}, \tag{56}$$

an irreducible fraction, for $k^{(\alpha-1)} + 1 \le j \le k^{(\alpha)}$, $1 \le \alpha \le \varepsilon$ (here $k^{(0)} = 0$ and $k^{(\varepsilon)} = n$). Since $k^{(\alpha)} - k^{(\alpha-1)}$ must be a multiple $n^{(\alpha)}\mu^{(\alpha)}$ of $\mu^{(\alpha)}$, we have $k^{(\alpha)} = n'\mu' + n''\mu'' + \ldots + n^{(\alpha)}\mu^{(\alpha)}$. If $k^{(\alpha)} \le n - m - 1 < k^{(\alpha+1)}$ and $\beta = n - m - 1 - k^{(\alpha)}$,

$$ht_m = n'm' + n''m'' + \ldots + n^{(\alpha)}m^{(\alpha)} - 2 + \frac{\beta m^{(\alpha+1)} + A_\beta^{(\alpha+1)}}{\mu^{(\alpha+1)}}, \tag{57}$$

where $A_\beta^{(\alpha+1)} = \mu^{(\alpha+1)} \varepsilon_{k^{(\alpha)}+\beta}$ is the remainder of the division of $-\beta m^{(\alpha+1)}$ by $\mu^{(\alpha+1)}$. For $\alpha = 1$, this shows that $t_{n-\beta-1} = 0$ unless $\frac{\beta m' + A'_\beta}{\mu'} \ge 2$. This inequality signifies that the quotient of $-\beta m'$ by μ' is ≤ -2 or that $\frac{\mu'}{m'} < \beta \le 2\frac{\mu'}{m'}$, the least possible value of β being $\beta' = E\left(\frac{\mu'}{m'} + 1\right)$ (integral part of $\frac{\mu'}{m'} + 1$). In addition, one must impose $\beta \le n-1$ and $\beta < k' = n'\mu'$ (condition neglected by Abel). Now if $\beta' > n-1$, $\frac{\mu'}{m'} + 1 \ge n$ and, since $\mu' \le n$, $\frac{\mu'}{m'}$ is equal to $\frac{1}{n}$ or to $\frac{1}{n-1}$, which imposes to χy to be of degree 1 with respect to x; in this case, $\int f(x, y)dx = \int Rdy$ with R rational in y, is algebraic and logarithmic in y. Sylow (*Œuvres*, t. II, p. 298) observes that the least possible value of β is still β' in the case in which

$\frac{\mu'}{m'} + 1 \leq n'\mu'$, with the only exceptions of χy of degree 1 with respect to x or $\chi y = y^2 + (Ax + B)y + Cx^2 + Dx + E$; in these cases, $\int f(x, y)dx$ may be reduced to the integral of a rational function and is expressible by algebraic and logarithmic functions. Finally, the abelian integrals leading to a constant in the right hand side of (55) are of the form

$$\int \frac{(t_0 + t_1 y + \ldots + t_{n-\beta'-1} y^{n-\beta'-1})dx}{\chi' y}$$

where the degree ht_m of each coefficient t_m is given by (57). Such a function involves a number of arbitrary constant equal to $\gamma = ht_0 + ht_1 + \ldots + ht_{n-\beta'-1} + n - \beta' = ht_0 + ht_1 + \ldots + ht_{n-2} + n - 1$. Using (57), Abel transforms this expression into

$$\begin{aligned}\gamma = &\frac{A'_0}{\mu'} + \frac{m' + A'_1}{\mu'} + \frac{2m' + A'_2}{\mu'} + \ldots + \frac{(n'\mu' - 1)m' + A'_{n'\mu'-1}}{\mu'} \\ &+ \frac{A''_0}{\mu''} + \frac{m'' + A''_1}{\mu''} + \frac{2m'' + A''_2}{\mu''} + \ldots + \frac{(n''\mu'' - 1)m'' + A''_{n''\mu''-1}}{\mu''} \\ &+ n'm'n''\mu'' \\ &+ \frac{A'''_0}{\mu'''} + \frac{m''' + A'''_1}{\mu'''} + \frac{2m''' + A'''_2}{\mu'''} + \ldots + \frac{(n'''\mu''' - 1)m''' + A'''_{n'''\mu'''-1}}{\mu'''} \\ &+ (n'm' + n''m'')n'''\mu''' \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots - n + 1.\end{aligned}$$

Since $A_0^{(\alpha)} + A_1^{(\alpha)} + \ldots + A^{(\alpha)}_{n^{(\alpha)}\mu^{(\alpha)}-1} = n^{(\alpha)} \frac{\mu^{(\alpha)}(\mu^{(\alpha)}-1)}{2}$ and $n = n'\mu' + n''\mu'' + \ldots + n^{(\varepsilon)}\mu^{(\varepsilon)}$, this finally gives

$$\begin{aligned}\gamma = &n'\mu'\frac{m'n' - 1}{2} + n''\mu''\left(m'n' + \frac{m''n'' - 1}{2}\right) \\ &+ n'''\mu'''\left(m'n' + m''n'' + \frac{m'''n''' - 1}{2}\right) + \ldots \\ &+ n^{(\varepsilon)}\mu^{(\varepsilon)}\left(m'n' + m''n'' + \ldots + m^{(\varepsilon-1)}n^{(\varepsilon-1)} + \frac{m^{(\varepsilon)}n^{(\varepsilon)} - 1}{2}\right) \\ &- \frac{n'(m' + 1)}{2} - \frac{n''(m'' + 1)}{2} - \ldots - \frac{n^{(\varepsilon)}(m^{(\varepsilon)} + 1)}{2} + 1. \qquad (58)\end{aligned}$$

Abel indicates some particular cases, first the case in which $\varepsilon = 1$ and

$$\gamma = n'\mu'\frac{m'n' - 1}{2} - n'\frac{m' + 1}{2} + 1;$$

more particularly, if in addition $\mu' = n$, one has $n' = 1$ and $\gamma = (n - 1)\frac{m'-1}{2}$. In the second particular case, $\mu' = \mu'' = \ldots = \mu^{(\varepsilon)} = 1 = n' = n'' = \ldots = n^{(\varepsilon)}$ and $\varepsilon = n$, thus

$$\gamma = (n-1)m' + (n-2)m'' + \ldots + 2m^{(n-2)} + m^{(n-1)} - n + 1$$

where $m^{(k)} = hy^{(k)}$ $(1 \le k \le n)$; when $hy^{(k)} = hy'$ for $1 \le k \le n-1$, this gives

$$\gamma = (n-1)\left(\frac{nhy'}{2} - 1\right).$$

Abel finally explains that the result remains true for the integrals of the form $\int \frac{f_1(x,y)dx}{\chi' y}$ even when $F_0 x$ is not constant, provided that $\frac{f_1(x,y)}{\chi' y}$ be finite whenever x is replaced by a root β of $F_0 x$ and y by the corresponding value B. In the final notes (*Œuvres*, t. II, p. 298), Sylow says under which precise conditions the number γ determined by Abel coincides with the *genus* p later defined by Riemann: the only multiple points of the curve defined by the equation $\chi y = 0$ must be at infinity in the directions of the axes and no two expansions of the $y^{(k)}$ in decreasing powers of x may begin by the same term.

As we have said, if there are α indeterminate coefficients $a, a', a'', \ldots$ in θy, one may choose arbitrarily α couple (x_j, y_j) of common roots to $\chi y = 0$ and $\theta y = 0$ and determine $a, a', a'', \ldots$ by the linear system $\theta y_j = 0$, $1 \le j \le \alpha$. If some couple (x_j, y_j) has a multiplicity k, one must replace the equation $\theta y_j = 0$ by $\theta y_j = \frac{d\theta y_j}{dx_j} = \ldots = \frac{d^{k-1}\theta y_j}{dx_j^{k-1}} = 0$. We get $a, a', a'', \ldots$ as rational functions of the (x_j, y_j) and we may substitute these functions in Fx. Abel (§ 6–7, p. 170–180) writes $Fx = B(x-x_1)(x-x_2)\ldots(x-x_\alpha)F^{(1)}x$ where $F^{(1)}x$ is a polynomial of degree $\mu - \alpha$ with coefficients rational in the (x_j, y_j) and $\psi x = \int f(x,y)dx$. According to (53),

$$\psi_1 x_1 + \psi_2 x_2 + \ldots + \psi_\alpha x_\alpha = v - (\psi_{\alpha+1} x_{\alpha+1} + \ldots + \psi_\mu x_\mu)$$

where $x_{\alpha+1}, \ldots, x_\mu$ are the roots of $F^{(1)}x = 0$, so algebraic functions of $x_1, \ldots, x_\alpha$, and v is an algebraic and logarithmic function.

Now α is of the form $hq_0 + hq_1 + \ldots + hq_{n-1} + n - 1 - hF_0x + A$ with $0 \le A \le hF_0x$ and $\mu = hr - hF_0x = h\theta y' + h\theta y'' + \ldots + h\theta y^{(n)} - hF_0x$. Thus

$$\mu - \alpha = h\theta y' + h\theta y'' + \ldots + h\theta y^{(n)} - (hq_0 + hq_1 + \ldots + hq_{n-1}) - n + 1 - A.$$

For any m, $h\theta y \ge h(q_m y^m) = hq_m + mhy$ and, according to (56),

$$h\theta y^{(j)} \ge hq_m + m\frac{m^{(\ell)}}{\mu^{(\ell)}} \quad \text{when } k^{(\ell-1)} + 1 \le j \le k^{(\ell)}. \tag{59}$$

Let us suppose that the maximum value of $h(q_m y^{(j)m})$ for $n - k^{(\ell)} \le m \le n - k^{(\ell-1)} - 1$ and $k^{(\ell-1)} + 1 \le j \le k^{(\ell)}$ is obtained for $m = \rho_\ell$:

$$hq_{\rho_\ell} + \rho_\ell \frac{m^{(\ell)}}{\mu^{(\ell)}} \ge hq_{n-\beta-1} + (n-\beta-1)\frac{m^{(\ell)}}{\mu^{(\ell)}} \quad \text{or}$$

$$hq_{\rho_\ell} - hq_{n-\beta-1} \ge (n-\beta-1-\rho_\ell)\frac{m^{(\ell)}}{\mu^{(\ell)}}$$

for $k^{(\ell-1)} \le \beta \le k^{(\ell)}-1$. Thus $hq_{\rho_\ell} - hq_{n-\beta-1} = (n-\beta-1-\rho_\ell)\frac{m^{(\ell)}}{\mu^{(\ell)}} + \varepsilon_\beta^{(\ell)} + A_\beta^{(\ell)}$ where $\varepsilon_\beta^{(\ell)}$ is a natural integer and $0 \le A_\beta^{(\ell)} < 1$. The sum of these equations for ℓ fixed and β variable gives

$$\begin{aligned}
n^{(\ell)}\mu^{(\ell)}&\left(hq_{\rho_\ell} + \rho_\ell\frac{m^{(\ell)}}{\mu^{(\ell)}}\right)\\
&= \frac{1}{2}(2n - k^{(\ell)} - k^{(\ell-1)} - 1)n^{(\ell)}\mu^{(\ell)} + A_0^{(\ell)} + A_1^{(\ell)} + \ldots + A_{n^{(\ell)}\mu^{(\ell)}-1}^{(\ell)}\\
&\quad + \varepsilon_0^{(\ell)} + \varepsilon_1^{(\ell)} + \ldots + \varepsilon_{n^{(\ell)}\mu^{(\ell)}-1}^{(\ell)} + hq_{n-1-k^{(\ell-1)}} + \ldots + hq_{n-k^{(\ell)}}\\
&= \frac{1}{2}(2n - k^{(\ell)} - k^{(\ell-1)} - 1)n^{(\ell)}\mu^{(\ell)}\\
&\quad + \frac{1}{2}n^{(\ell)}(\mu^{(\ell)}-1) + C_\ell + hq_{n-1-k^{(\ell-1)}} + \ldots + hq_{n-k^{(\ell)}}
\end{aligned}$$

where $C_\ell = \varepsilon_0^{(\ell)} + \varepsilon_1^{(\ell)} + \ldots + \varepsilon_{n^{(\ell)}\mu^{(\ell)}-1}^{(\ell)}$. Let us write the inequalities (59) for $k^{(\ell-1)}+1 \le j \le k^{(\ell)}$, $m = \rho_\ell$ and then sum up all these inequalities for ℓ variable. This gives

$$\begin{aligned}
h\theta y' + h\theta y'' + \ldots + h\theta y^{(n)}&\\
\ge hq_{n-1} + hq_{n-2} + \ldots + hq_0&\\
+ \sum_{\ell=1}^{\varepsilon}\left(\frac{1}{2}(2n - k^{(\ell)} - k^{(\ell-1)} - 1)n^{(\ell)}\mu^{(\ell)} + \frac{1}{2}n^{(\ell)}(\mu^{(\ell)}-1) + C_\ell\right)&
\end{aligned}$$

or $h\theta y' + h\theta y'' + \ldots + h\theta y^{(n)} - (hq_0 + hq_1 + \ldots + hq_{n-1}) \ge \gamma' + C_1 + C_2 + \ldots + C_\varepsilon$ where

$$\begin{aligned}
\gamma' = &\, n'm'\left(\frac{n'\mu'-1}{2} + n''\mu'' + n'''\mu''' + \ldots + n^{(\varepsilon)}\mu^{(\varepsilon)}\right) + n'\frac{\mu'-1}{2}\\
&+ n''m''\left(\frac{n''\mu''-1}{2} + n'''\mu''' + n''''\mu'''' + \ldots + n^{(\varepsilon)}\mu^{(\varepsilon)}\right) + n''\frac{\mu''-1}{2}\\
&+ \ldots\\
&+ n^{(\varepsilon-1)}m^{(\varepsilon-1)}\left(\frac{n^{(\varepsilon-1)}\mu^{(\varepsilon-1)}-1}{2} + n^{(\varepsilon)}\mu^{(\varepsilon)}\right) + n^{(\varepsilon-1)}\frac{\mu^{(\varepsilon-1)}-1}{2}\\
&+ n^{(\varepsilon)}m^{(\varepsilon)}\frac{n^{(\varepsilon)}\mu^{(\varepsilon)}-1}{2} + n^{(\varepsilon)}\frac{\mu^{(\varepsilon)}-1}{2}.
\end{aligned}$$

We finally obtain $\mu - \alpha \ge \gamma' - n + 1 - A + C_1 + C_2 + \ldots + C_\varepsilon$ and we remark that, according to (58),

$$\gamma' - n + 1 = \gamma,$$

so that $\mu - \alpha \ge \gamma - A + C_1 + C_2 + \ldots + C_\varepsilon$.

As Abel notes it, $\mu - \alpha = \gamma - A$ when $C_1 + C_2 + \ldots + C_\varepsilon = 0$ and, for each ℓ,

$$h\theta y^{k^{(\ell)}} = hq_{\rho_\ell} + \rho_\ell \frac{m^{(\ell)}}{\mu^{(\ell)}}. \tag{60}$$

He shows that, for a convenient choice of θy, these conditions are realised. The first one signifies that $\varepsilon_\beta^{(\ell)} = 0$ for $k^{(\ell-1)} \leq \beta \leq k^{(\ell)} - 1$ and $1 \leq \ell \leq \varepsilon$ or that

$$hq_{n-\beta-1} = hq_{\rho_\ell} - (n - \beta - 1 - \rho_\ell)\frac{m^{(\ell)}}{\mu^{(\ell)}} - A_\beta^{(\ell)} \tag{61}$$

for $k^{(\ell-1)} \leq \beta \leq k^{(\ell)} - 1$. The degrees hq_m will then be definite if we know the hq_{ρ_ℓ} and we have by (60),

$$hq_{\rho_\ell} + \rho_\ell \frac{m^{(\ell)}}{\mu^{(\ell)}} \geq hq_{\rho_\alpha} + \rho_\alpha \frac{m^{(\ell)}}{\mu^{(\ell)}} \tag{62}$$

for any ℓ and any α. Abel puts $\frac{m^{(\ell)}}{\mu^{(\ell)}} = \sigma_\ell$ and deduces from the preceding inequality

$$(\rho_{\ell-1} - \rho_\ell)\sigma_\ell \leq hq_{\rho_\ell} - hq_{\rho_{\ell-1}} \leq (\rho_{\ell-1} - \rho_\ell)\sigma_{\ell-1}.$$

Thus $hq_{\rho_\ell} - hq_{\rho_{\ell-1}} = (\rho_{\ell-1} - \rho_\ell)(\theta_{\ell-1}\sigma_{\ell-1} + (1 - \theta_{\ell-1})\sigma_\ell)$ where $0 \leq \theta_{\ell-1} \leq 1$, and $hq_{\rho_\ell} = hq_{\rho_1} + (\rho_1 - \rho_2)(\theta_1\sigma_1 + (1-\theta_1)\sigma_2) + (\rho_2 - \rho_3)(\theta_2\sigma_2 + (1-\theta_2)\sigma_3) + \ldots + (\rho_{\ell-1} - \rho_\ell)(\theta_{\ell-1}\sigma_{\ell-1} + (1-\theta_{\ell-1})\sigma_\ell)$. Inversely, for any choice of hq_{ρ_1} and of the θ_α (between 0 and 1), these values of hq_{ρ_ℓ} verify the inequalities (62). It is then possible, using (61) and some work, to prove that (60) is verified; a narrower limitation is imposed to the θ_α.

All this discussion was made in the hypothesis that the only condition limiting the indetermination of the coefficients of the q_m was that the polynomial $F_0 x$ divides the resultant r. When more conditions are imposed to limit the number α of the indeterminate coefficients $a, a', a'', \ldots$, the minimum value of $\mu - \alpha$ may be of the form $\gamma - A - B < \gamma - A$. In the final notes (*Œuvres*, t. II, p. 299–300), Sylow explains that A is the reduction due to the presence of singularities at a finite distance on the curve $\chi y = 0$ and that the additional reduction B is due to the eventual coincidence of the initial terms in some of the $y^{(k)}$. Moreover he explains how Abel's formula (58) may lead to a computation of A.

In the following paragraph 8 (p. 181–185), Abel explicitly deals with the case where χy is of degree $n = 13$ in y, the degrees in x of the coefficients p_m being 2 for $m = 0, 2, 8$; 3 for $m = 1, 3, 6, 9$; 4 for $m = 4, 7, 10$; 5 for $m = 5$ and 1 for $m = 11, 12$. He determines the exponents $hy^{(k)}$ by a method similar to that of the Newton polygon and finds $hy' = hy'' = hy''' = \frac{m'}{\mu'} = \frac{4}{3}$, $n' = 1$; $hy^{(4)} = hy^{(5)} = hy^{(6)} = hy^{(7)} = hy^{(8)} = \frac{m''}{\mu''} = \frac{1}{5}$, $n'' = 1$; $hy^{(9)} = hy^{(10)} = hy^{(11)} = hy^{(12)} = \frac{m'''}{\mu'''} = \frac{-1}{2}$, $n''' = 2$; $hy^{(13)} = \frac{m''''}{\mu''''} = -1$, $n'''' = 1$. These values give $\gamma = 38$ and the limitations $10 \leq \rho_1 \leq 12$, $5 \leq \rho_2 \leq 9$, $1 \leq \rho_3 \leq 4$ and $\rho_4 = 0$. Choosing for instance $\rho_1 = 11$, $\rho_2 = 6$, $\rho_3 = 4$, he finds $A'_0 = \frac{2}{3}$, $A'_2 = \frac{1}{3}$, $A''_3 = \frac{2}{5}$, $A''_4 = \frac{3}{5}$, $A''_5 = \frac{4}{5}$, $A''_7 = \frac{1}{5}$, $A'''_9 = \frac{1}{2}$, $A'''_{10} = 0$, $A'''_{11} = \frac{1}{2}$ and then $\frac{12}{85} \leq \theta_1 \leq \frac{8}{17}$, $\frac{5}{14} \leq \theta_2 \leq 1$ and $\frac{1}{2} \leq \theta_3 \leq 1$. The values of the differences

$$hq_6 - hq_{11}, \quad hq_4 - hq_6, \quad hq_0 - hq_4$$

are correspondingly limited and they may be: $hq_6 - hq_{11} = 2, 3$; $hq_4 - hq_6 = 0$; $hq_0 - hq_4 = -3, -2$. It is now possible to determine all the degrees hq_m knowing $\theta = hq_{12}$. The possible values of α are $13\theta + 47$, $13\theta + 48$, $13\theta + 57$ or $13\theta + 58$. The corresponding values of μ are $13\theta + 85$, $13\theta + 86$, $13\theta + 95$ and $13\theta + 96$. Thus $\mu - \alpha = 38 = \gamma$ for every choice.

Then (§ 9, p. 185–188) Abel extends his relation (53) in the form

$$h_1\psi_1 x_1 + h_2\psi_2 x_2 + \ldots + h_\alpha \psi_\alpha x_\alpha = v$$

where the coefficients $h_1, h_2, \ldots, h_\alpha$ are rational numbers. In the paragraph 10 (p. 188–211), he deals with the case in which $\chi y = y^n + p_0$ where p_0 is a polynomial in x and the integral $\psi x = \int \frac{f_3 x \cdot dx}{y^m f_2 x}$ where $f_2 x$ and $f_3 x$ are polynomials in x. If $-p_0 = r_1^{\mu_1} r_2^{\mu_2} \ldots r_\varepsilon^{\mu_\varepsilon}$ where the polynomials $r_1, r_2, \ldots, r_\varepsilon$ are squarefree and relatively prime by pairs, let us put with Abel $R = r_1^{\frac{\mu_1}{n}} r_2^{\frac{\mu_2}{n}} \ldots r_\varepsilon^{\frac{\mu_\varepsilon}{n}}$, so that the determinations of y are $y^{(k)} = \omega^{k-1} R (1 \leq k \leq n)$, ω being a primitive n-th root of 1. The determinations of the integral ψx are of the form $\omega^{-em} \int \frac{f_3 x \cdot dx}{R^m f_2 x}$ where e is an integer and (53) takes the form

$$\omega^{-e_1 m}\psi x_1 + \omega^{-e_2 m}\psi x_2 + \ldots + \omega^{-e_\mu m}\psi x_\mu = C - \prod \frac{\varphi_2 x}{f_2 x} + \sum{}' \frac{1}{\Gamma \nu} \frac{d^{\nu-1}}{dx^{\nu-1}} \left(\frac{F_2 x \cdot \varphi_2 x}{\vartheta x} \right) \tag{63}$$

where

$$\varphi_2 x = \frac{f_3 x}{R^m} (\log \theta R + \omega^{-m} \log \theta(\omega R) + \omega^{-2m} \log \theta(\omega^2 R) + \ldots + \omega^{-(n-1)m} \log \theta(\omega^{n-1} R)). \tag{64}$$

Let us first suppose that all the coefficients in $q_0, q_1, \ldots, q_{n-1}$ are indeterminate, so that $\alpha = hq_0 + hq_1 + \ldots + hq_{n-1} + n - 1$. In our case, $hy' = hy'' = \ldots = hy^{(n)} = \frac{m'}{\mu'}$. We have $\varepsilon = 1$ and $n = n'\mu' = k'$. Let us determine the minimum value of $\mu - \alpha$. According to the relation (61),

$$hq_m = hq_{\rho_1} + (\rho_1 - m)\frac{m'}{\mu'} - A'_m \tag{65}$$

with $0 \leq A'_m < 1$. Here the number μ is

$$hr = nhq_{\rho_1} + n'm'\rho_1 \tag{66}$$

and, according to (58), $\mu - \alpha = \gamma = n'\mu' \frac{n'm'-1}{2} - n' \frac{m'+1}{2} + 1 = \frac{n-1}{2} nhR - \frac{n+n'}{2} + 1$. But this value can be lowered by a more convenient choice of θy. For $1 \leq m \leq \varepsilon$ and $0 \leq \pi \leq n - 1$, Abel puts $\theta_m = E\frac{\mu_m}{n} + E\frac{2\mu_m}{n} + \ldots + E\frac{(n-1)\mu_m}{n}$ and $\delta_{m,\pi} = \theta_m - E\left(\frac{\pi\mu_m}{n} - \frac{\alpha_m}{n}\right)$ where E denotes the integral part of the following fraction and the α_m are natural integers. He takes the coefficients q_π of θy of the form $q_\pi = v_\pi r_1^{\delta_{1,\pi}} r_2^{\delta_{2,\pi}} \dots r_\varepsilon^{\delta_{\varepsilon,\pi}}$ where the v_π are polynomials in x. Then $q_\pi R^\pi = v_\pi r_1^{\theta_1+\frac{\alpha_1}{n}} r_2^{\theta_2+\frac{\alpha_2}{n}} \dots r_\varepsilon^{\theta_\varepsilon+\frac{\alpha_\varepsilon}{n}} R^{(\pi)}$ where $R^{(\pi)} = r_1^{k_{1,\pi}} r_2^{k_{2,\pi}} \dots r_\varepsilon^{k_{\varepsilon,\pi}}$, $k_{m,\pi} = \varepsilon\frac{\pi\mu_m - \alpha_m}{n}$ (ε denotes the excess of the following fraction over its integral part), and

$$\theta y^{(e+1)} = \theta'(x, e) r_1^{\theta_1+\frac{\alpha_1}{n}} r_2^{\theta_2+\frac{\alpha_2}{n}} \dots r_\varepsilon^{\theta_\varepsilon+\frac{\alpha_\varepsilon}{n}}$$

where $\theta'(x, e) = v_0 R^{(0)} + \omega^e v_1 R^{(1)} + \omega^{2e} v_2 R^{(2)} + \dots + \omega^{(n-1)e} v_{n-1} R^{(n-1)}$, $0 \le e \le n-1$. This gives $F_0 x = r_1^{n\theta_1+\alpha_1} r_2^{n\theta_2+\alpha_2} \dots r_\varepsilon^{n\theta_\varepsilon+\alpha_\varepsilon}$ and $Fx = \theta'(x, 0)\theta'(x, 1) \dots \theta'(x, n-1)$. Now (54) takes the form $Rx = \sum \frac{f_3 x}{y^m} \frac{r \cdot \delta\theta y}{\theta y} = F_0 x \sum \frac{f_3 x}{y^{(e+1)m}} \frac{Fx \cdot \delta\theta'(x,e)}{\theta'(x,e)}$ and $\frac{f_3 x}{y^m} = \frac{fx}{s_m}$ where

$$fx = f_3 x \cdot r_1^{-E\frac{m\mu_1}{n}} r_2^{-E\frac{m\mu_2}{n}} \dots r_\varepsilon^{-E\frac{m\mu_\varepsilon}{n}} \text{ and } s_m = r_1^{\varepsilon\frac{m\mu_1}{n}} r_2^{\varepsilon\frac{m\mu_2}{n}} \dots r_\varepsilon^{\varepsilon\frac{m\mu_\varepsilon}{n}}.$$

Thus

$$\begin{aligned} Rx = \frac{F_0 x \cdot fx}{s_m} \biggl(& \frac{Fx}{\theta'(x, 0)} \delta\theta'(x, 0) + \omega^{-m} \frac{Fx}{\theta'(x, 1)} \delta\theta'(x, 1) \\ & + \omega^{-2m} \frac{Fx}{\theta'(x, 2)} \delta\theta(x, 2) + \dots \\ & + \omega^{-(n-1)m} \frac{Fx}{\theta'(x, n-1)} \delta\theta'(x, n-1) \biggr). \end{aligned}$$

Since the $\frac{Fx}{\theta'(x,e)} \delta\theta'(x, e)$ are polynomial in x, $R^{(0)}, R^{(1)}, \dots, R^{(n-1)}$, so linear combinations of the s_m with coefficients polynomial in x, it results that $F_0 x$ divides Rx: $Rx = F_0 x \cdot R_1 x$. Now one sees that $F_2 x = 1$, $\theta_1 x = f_2 x$ and that (64) takes the form

$$\begin{aligned} \varphi_2 x = \frac{fx}{s_m} (& \log\theta'(x, 0) + \omega^{-m} \log\theta'(x, 1) + \omega^{-2m} \log\theta'(x, 2) \\ & + \dots + \omega^{-(n-1)m} \log\theta'(x, n-1)). \end{aligned}$$

Here

$$\begin{aligned} \mu &= hr - hF_0 x \\ &= nhq_{\rho_1} + n'm'\rho_1 - ((n\theta_1 + \alpha_1)hr_1 + (n\theta_2 + \alpha_2)hr_2 + \dots + (n\theta_\varepsilon + \alpha_\varepsilon)hr_\varepsilon) \end{aligned}$$

where $n'm' = nhR = \mu_1 hr_1 + \mu_2 hr_2 + \dots + \mu_\varepsilon hr_\varepsilon$. Thus, putting ρ for ρ_1,

$$\begin{aligned} \mu = nhq_\rho &+ (\mu_1\rho - n\theta_1 - \alpha_1)hr_1 + (\mu_2\rho - n\theta_2 - \alpha_2)hr_2 \\ &+ \dots + (\mu_\varepsilon\rho - n\theta_\varepsilon - \alpha_\varepsilon)hr_\varepsilon \\ = nhv_\rho &+ (n\delta_{1,\rho} - n\theta_1 + \rho\mu_1 - \alpha_1)hr_1 + (n\delta_{\mathrm{ys}\rho} - n\theta_2 + \rho\mu_2 - \alpha_2)hr_2 \\ &+ \dots + (n\delta_{\varepsilon,\rho} - n\theta_\varepsilon + \rho\mu_\varepsilon - \alpha_\varepsilon)hr_\varepsilon \\ &= nhv_\rho + n\varepsilon\frac{\rho\mu_1 - \alpha_1}{n} hr_1 + n\varepsilon\frac{\rho\mu_2 - \alpha_2}{n} hr_2 + \dots + n\varepsilon\frac{\rho\mu_\varepsilon - \alpha_\varepsilon}{n} hr_\varepsilon. \end{aligned}$$

On the other hand

$$\begin{aligned}\alpha &= hv_0 + hv_1 + \ldots + v_{n-1} + n - 1\\ &= hq_0 + hq_1 + \ldots + q_{n-1} + n - 1 - \sum_m (\delta_{m,0} + \delta_{m,1} + \ldots + \delta_{m,n-1})hr_m\\ &= n\left(hv_\rho + \sum_m \delta_{m,\rho} hr_m\right) + \left(n\rho - \frac{n(n-1)}{2}\right)\frac{m'}{\mu'}\\ &\quad - \frac{n'(\mu'-1)}{2} - \sum_m (\delta_{m,0} + \delta_{m,1} + \ldots + \delta_{m,n-1})hr_m.\end{aligned}$$

Abel computes $\delta_{m,0} + \delta_{m,1} + \ldots + \delta_{m,n-1} = \alpha_m + (n-1)\theta_m$ and finally gets

$$\begin{aligned}\alpha = nhv_\rho &+ \left(n\varepsilon\frac{\rho\mu_1 - \alpha_1}{n} - \frac{n-k_1}{2}\right)hr_1 + \left(n\varepsilon\frac{\rho\mu_2 - \alpha_2}{n} - \frac{n-k_2}{2}\right)hr_2\\ &+ \ldots + \left(n\varepsilon\frac{\rho\mu_\varepsilon - \alpha_\varepsilon}{n} - \frac{n-k_\varepsilon}{2}\right)hr_\varepsilon - 1 + \frac{n+n'}{2}\end{aligned}$$

where k_m is the g.c.d. of μ_m and n . This gives

$$\mu - \alpha = \frac{n-k_1}{2}hr_1 + \frac{n-k_2}{2}hr_2 + \ldots + \frac{n-k_\varepsilon}{2}hr_\varepsilon + 1 - \frac{n+n'}{2} = \theta \tag{67}$$

independent of $\rho, \alpha_1, \alpha_2, \ldots, \alpha_\varepsilon$ and we have $\mu = nhv_\rho + nhR^{(\rho)}$ (cf. (66)). The degrees hv_m are determined from $hq_m = \delta_{1,m}hr_1 + \delta_{2,m}hr_2 + \ldots + \delta_{\varepsilon,m}hr_\varepsilon + v_m$ and (65) which give

$$\begin{aligned}hq_m &= hv_\rho + (\rho - m)\frac{m'}{\mu'} + (\delta_{1,\rho} - \delta_{1,m})hr_1 + (\delta_{2,\rho} - \delta_{2,m})hr_2\\ &\quad + \ldots + (\delta_{\varepsilon,\rho} - \delta_{\varepsilon,m})hr_\varepsilon - A'_m\\ &= hv_\rho + E((k_{1,\rho} - k_{1,m})hr_1 + (k_{2,\rho} - k_{2,m})hr_2\\ &\quad + \ldots + (k_{\varepsilon,\rho} - k_{\varepsilon,m})hr_\varepsilon)\\ &= hv_\rho + Eh\frac{R^{(\rho)}}{R^{(m)}}.\end{aligned} \tag{68}$$

Abel adopts new notations: $x_{\alpha+1} = z_1$, $x_{\alpha+2} = z_2, \ldots, x_\mu = z_\theta$; $e_{\alpha+1} = \varepsilon_1, e_{\alpha+2} = \varepsilon_2, \ldots, e_\mu = \varepsilon_\mu$; $\omega^{-e_\mu} = \omega_\mu$ and $\omega^{-\varepsilon_\mu} = \pi_\mu$ and he rewrites (63) in the form

$$\begin{aligned}&\omega_1^m\psi x_1 + \omega_2^m\psi x_2 + \ldots + \omega_\alpha^m\psi x_\alpha + \pi_1^m\psi z_1 + \pi_2^m\psi z_2 + \ldots + \pi_\theta^m\psi z_\theta \\ &\quad = C - \prod\frac{fx\varphi x}{s_m(x)f_2x} + \sum{}' \frac{1}{\Gamma\nu}\frac{d^{\nu-1}}{dx^{\nu-1}}\left(\frac{fx\cdot\varphi x}{s_m(x)\vartheta x}\right)\end{aligned} \tag{69}$$

where $\theta_1(x) = f_2x = A(x-\beta_1)^{\nu_1}(x-\beta_2)^{\nu_2}\ldots$, fx is an arbitrary polynomial,

$$\begin{aligned}\varphi x = \log\theta'(x,0) &+ \omega^{-m}\log\theta'(x,1) + \omega^{-2m}\log\theta'(x,2)\\ &+ \ldots + \omega^{-(n-1)m}\log\theta'(x,n-1)\end{aligned}$$

and $\psi x = \int \frac{fxdx}{f_2x \cdot s_m(x)}$. Here, $x_1, x_2, \ldots, x_\alpha$ are considered as independent variables and $z_1, z_2, \ldots, z_\theta$ are the roots of the equation $\frac{\theta'(z,0)\theta'(z,1)\ldots\theta'(z,n-1)}{(z-x_1)(z-x_2)\ldots(z-x_\alpha)} = 0$. The coefficients $a, a', a'', \ldots$ are determined by the equations $\theta'(x_1, e_1) = \theta'(x_2, e_2) = \ldots = \theta'(x_\alpha, e_\alpha) = 0$ and the numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\theta$ by $\theta'(z_1, \varepsilon_1) = \theta'(z_2, \varepsilon_2) = \ldots = \theta'(z_\theta, \varepsilon_\theta) = 0$.

Some particular cases are explicited by Abel, first the case in which $f_2x = (x-\beta)^\nu$ with, for instance, $\nu = 1$ or 0. In this last case the right hand side of (69) reduces to

$$C - \prod \frac{fx \cdot \varphi x}{s_m(x) f_2x}$$

which is constant when $hfx \le -E(-hs_m(x)) - 2$.

When $n = 1$, there is only one $s_m = s_0 = 1$ and $\psi x = \int \frac{fx \cdot dx}{f_2x}$. Then $R^{(0)} = 1$, $\theta'(x, 0) = v_0$ and $\varphi x = \log v_0$. The relation (68) takes the form

$$\psi x_1 + \psi x_2 + \ldots + \psi x_\alpha + \psi z_1 + \psi z_2 + \ldots + \psi z_\theta$$
$$= C - \prod \frac{fx}{f_2x} \log v_0 + \sum{}' \frac{1}{\Gamma\nu} \frac{d^{\nu-1}}{dx^{\nu-1}} \left(\frac{fx}{\vartheta x} \log v_0 \right)$$

where $v_0(x) = a(x-x_1)(x-x_2)\ldots(x-x_\alpha)(x-z_1)(x-z_2)\ldots(x-z_\theta)$, but it is possible to make $\theta = 0$ in (67). For $\alpha = 1$, one finds the known integration of rational differential forms.

When $n = 2$ and $R = r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}$, take $\alpha_1 = 1$ and $\alpha_2 = 0$. Then $s_0 = 1$, $s_1 = (r_1 r_2)^{\frac{1}{2}}$, $R^{(0)} = r_1^{\frac{1}{2}}$, $R^{(1)} = r_2^{\frac{1}{2}}$, $\theta'(x, 0) = v_0 r_1^{\frac{1}{2}} + v_1 r_2^{\frac{1}{2}}$, $\theta'(x, 1) = v_0 r_1^{\frac{1}{2}} - v_1 r_2^{\frac{1}{2}}$ and $\omega = -1$. For $m = 1$, we find $\varphi x = \log \frac{v_0 r_1^{\frac{1}{2}} + v_1 r_2^{\frac{1}{2}}}{v_0 r_1^{\frac{1}{2}} - v_1 r_2^{\frac{1}{2}}}$ and, writing $\varphi_0 x$ and $\varphi_1 x$ respectively for r_1 and r_2, (69) takes the form

$$\sum \omega\psi x + \sum \pi\psi z = C - \prod \frac{fx}{f_2x\sqrt{\varphi_0 x \varphi_1 x}} \log \frac{v_0\sqrt{\varphi_0 x} + v_1\sqrt{\varphi_1 x}}{v_0\sqrt{\varphi_0 x} - v_1\sqrt{\varphi_1 x}}$$
$$+ \sum{}' \frac{1}{\Gamma\nu} \frac{d^{\nu-1}}{dx^{\nu-1}} \frac{fx}{\vartheta x\sqrt{\varphi_0 x \varphi_1 x}} \log \frac{v_0\sqrt{\varphi_0 x} + v_1\sqrt{\varphi_1 x}}{v_0\sqrt{\varphi_0 x} - v_1\sqrt{\varphi_1 x}}$$

where $\psi x = \int \frac{fx \cdot dx}{f_2x\sqrt{\varphi_0 x \varphi_1 x}}$, v_0 and v_1 are determined by the equations $v_0\sqrt{\varphi_0 x_1} + \omega_1 v_1 \sqrt{\varphi_1 x_1} = v_0\sqrt{\varphi_0 x_2} + \omega_2 v_1\sqrt{\varphi_1 x_2} = \ldots = 0$ and $z_1, z_2, \ldots, z_\theta$ by $\frac{(v_0(z))^2\varphi_0 x - (v_1(z))^2\varphi_1 x}{(z-x_1)(z-x_2)\ldots(z-x_\alpha)} = 0$. The signs π_k are given by $\pi_k = -\frac{v_0(z_k)\sqrt{\varphi_0 z_k}}{v_1(z_k)\sqrt{\varphi_1 z_k}}$. We have $k_1 = k_2 = 1$, $\theta = \frac{1}{2}hr_1 + \frac{1}{2}hr_2 - \frac{n'}{2} = \frac{1}{2}(h(r_1r_2) - n')$ where n' is the g.c.d. of 2 and $h(r_1r_2)$. Thus $\theta = m - 1$ for $h(\varphi_0 x \varphi_1 x) = 2m - 1$ or $2m$. Taking $\rho = 1$, we have by (68)

$$hv_0 = v_1 + E\frac{1}{2}(h\varphi_1 x - h\varphi_0 x) = \begin{cases} hv_1 + \frac{1}{2}(h\varphi_1 x - h\varphi_0 x) - \frac{1}{2} \\ hv_1 + \frac{1}{2}(h\varphi_1 x - h\varphi_0 x) \end{cases}$$

depending on whether $h(\varphi_0 x \cdot \varphi_1 x)$ is odd or even. When $m = 1$, $\theta = 0$ and $\psi x = \int \frac{fx \cdot dx}{f_2 x \sqrt{R}}$ where R is a polynomial of degree 1 or 2. This integral is an algebraic and logarithmic function and Abel explicits the computation, taking $\varphi_0 x = \varepsilon_0 x + \delta_0$, $\varphi_1 x = \varepsilon_1 x + \delta_1$, $f_2 x = (x - \beta)^\nu$, $v_1 = 1$ and $v_0 = a$. When $m = 2$, $\theta = 1$ and $h(\varphi_0 x \cdot \varphi_1 x) = 3$ or 4 so that ψx is an elliptic integral. The relation (69) takes the form $\omega_1 \psi x_1 + \omega_2 \psi x_2 + \ldots + \omega_\alpha \psi x_\alpha = v - \pi_1 \psi z_1$ where v is algebraic and logarithmic. The product of the roots of the polynomial $(v_0 z)^2 \varphi_0 z - (v_1 z)^2 \varphi_1 z = A + \ldots + B z^{\alpha+1}$ is $x_1 x_2 \ldots x_\alpha z_1$, whence $z_1 = \frac{A}{B} \frac{(-1)^{\alpha+1}}{x_1 x_2 \ldots x_\alpha}$, where $\frac{A}{B}$ is a rational function of x_1, $x_2, \ldots, x_\alpha$, $\sqrt{\varphi_0 x_1}$, $\sqrt{\varphi_0 x_2}, \ldots, \sqrt{\varphi_0 x_\alpha}$, $\sqrt{\varphi_1 x_1}$, $\sqrt{\varphi_1 x_2}, \ldots, \sqrt{\varphi_1 x_\alpha}$. When

$$\varphi_0 x = 1, \varphi_1 x = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3, v_1 = 1 \text{ and } v_0 = a_0 + a_1 x,$$

we must write $v_0 x_1 = -\omega_1 \sqrt{\varphi_1 x_1}$, $v_0 x_2 = -\omega_2 \sqrt{\varphi_1 x_2}$, whence

$$a_0 = \frac{\omega_1 x_2 \sqrt{\varphi_1 x_1} - \omega_2 x_1 \sqrt{\varphi_1 x_2}}{x_1 - x_2}, \quad a_1 = \frac{\omega_2 \sqrt{\varphi_1 x_2} - \omega_1 \sqrt{\varphi_1 x_1}}{x_1 - x_2}.$$

Then $A = a_0^2 - \alpha_0$, $B = -\alpha_3$ and $z_1 = \frac{1}{\alpha_3 x_1 x_2} \left(\frac{x_2^2 \varphi_1 x_1 + x_1^2 \varphi_1 x_2 - 2\omega_1 \omega_2 x_1 x_2 \sqrt{\varphi_1 x_1 \varphi_1 x_2}}{(x_1 - x_2)^2} - \alpha_0 \right)$ which gives the addition theorem for elliptic integrals.

When $m = 3$, $\theta = 2$ and $h(\varphi_0 x \cdot \varphi_1 x) = 5$ or 6. Abel explains certain particular cases, for instance that in which $\psi x = \int \frac{(A_0 + A_1 x) dx}{\sqrt{\alpha_0 + \alpha_1 x + \ldots + \alpha_6 x^6}}$, which gives

$$\pm \psi x_1 \pm \psi x_2 \pm \ldots \pm \psi x_\alpha = \pm \psi z_1 \pm \psi z_2 + C,$$

where z_1, z_2 are the roots of a quadratic equation with coefficients rational in x_1, $x_2, \ldots, x_\alpha$, $\sqrt{R_1}$, $\sqrt{R_2}, \ldots, \sqrt{R_\alpha}$ (where R_k is the value of R corresponding to $x = x_k$). As we have said in our §1, Abel explained this result, with $\alpha = 3$, in a letter to Crelle.

The last example dealt with by Abel is not hyperelliptic for he takes $n = 3$, $R = r_1^{\frac{1}{3}} r_2^{\frac{2}{3}}$, $\alpha_1 = \alpha_2 = 0$. Then $s_0 = 1$, $s_1 = r_1^{\frac{1}{3}} r_2^{\frac{2}{3}}$, $s_2 = r_1^{\frac{2}{3}} r_2^{\frac{1}{3}}$, $R^{(0)} = s_0$, $R^{(1)} = s_1$, $R^{(2)} = s_2$ and

$$\theta'(x, 0) = v_0 + v_1 r_1^{\frac{1}{3}} r_2^{\frac{2}{3}} + v_2 r_1^{\frac{2}{3}} r_2^{\frac{1}{3}}, \theta'(x, 1) = v_0 + \omega v_1 r_1^{\frac{1}{3}} r_2^{\frac{2}{3}} + \omega^2 v_2 r_1^{\frac{2}{3}} r_2^{\frac{1}{3}},$$
$$\theta'(x, 2) = v_0 + \omega^2 v_1 r_1^{\frac{1}{3}} r_2^{\frac{2}{3}} + \omega v_2 r_1^{\frac{2}{3}} r_2^{\frac{1}{3}},$$

which give $Fx = \theta'(x, 0)\theta'(x, 1)\theta'(x, 2) = v_0^3 + v_1^3 r_1 r_2^2 + v_2^3 r_1^2 r_2 - 3 v_0 v_1 v_2 r_1 r_2$. Here

$$\theta = h r_1 + h r_2 + 1 - \frac{3 + n'}{2}$$

where n' is the g.c.d. of 3 and $r_1 + 2 h r_2$. Thus $\theta = h(\varphi_0 x \varphi_1 x) - 2$ if $h r_1 + 2 h r_2$ is divisible by 3 and $\theta = h(\varphi_0 x \varphi_1 x) - 1$ in the contrary case.

Since the french Academy did not give any news of his memoir, Abel decided to published his theorem for the particular case of hyperelliptic integrals in Crelle's

Journal (vol. 3, 1828, *Œuvres*, t. I, p. 444–456). It is this publication which inspired Jacobi for the formulation of the inversion problem (1832). In a letter to Legendre (14 March 1829), Jacobi said of Abel theorem that it was "perhaps the most important discovery of what the century in which we live has made in mathematics ... though only a work to come, in a may be distant future, may throw light on its full importance". The statement is the following: "Let φx be a polynomial in x, decomposed in two factors $\varphi_1 x, \varphi_2 x$ and let fx be another polynomial and $\psi x = \int \frac{fx \cdot dx}{(x-\alpha)\sqrt{\varphi x}}$ where α is an any constant quantity. Let us designate by $a_0, a_1, a_2, \ldots, c_0, c_1, c_2, \ldots$ arbitrary quantities of which at least one is variable. Then if one puts

$$(a_0 + a_1 x + \ldots + a_n x^n)^2 \varphi_1 x - (c_0 + c_1 x + \ldots + c_m x^m)^2 \varphi_2 x = A(x - x_1)(x - x_2)\ldots(x - x_\mu)$$

where A does not depend on x, I say that

$$\varepsilon_1 \psi x_1 + \varepsilon_2 \psi x_2 + \ldots + \varepsilon_\mu \psi x_\mu = -\frac{f\alpha}{\sqrt{\varphi\alpha}} \log \frac{(a_0 + a_1\alpha + \ldots + a_n\alpha^n)\sqrt{\varphi_1\alpha} + (c_0 + c_1\alpha + \ldots + c_m\alpha^m)\sqrt{\varphi_2\alpha}}{(a_0 + a_1\alpha + \ldots + a_n\alpha^n)\sqrt{\varphi_1\alpha} - (c_0 + c_1\alpha + \ldots + c_m\alpha^m)\sqrt{\varphi_2\alpha}} + r + C$$

where C is a constant quantity and r the coefficient of $\frac{1}{x}$ in the expansion of

$$\frac{fx}{(x-\alpha)\sqrt{\varphi x}} \log \frac{(a_0 + a_1 x + \ldots + a_n x^n)\sqrt{\varphi_1 x} + (c_0 + c_1 x + \ldots + c_m x^m)\sqrt{\varphi_2 x}}{(a_0 + a_1 x + \ldots + a_n x^n)\sqrt{\varphi_1 x} - (c_0 + c_1 x + \ldots + c_m x^m)\sqrt{\varphi_2 x}}$$

in decreasing powers of x. The quantities $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\mu$ are equal to $+1$ or to -1 and their values depend on those of the quantities $x_1, x_2, \ldots, x_\mu$."

Putting $\theta x = a_0 + a_1 x + \ldots + a_n x^n$, $\theta_1 x = c_0 + c_1 x + \ldots + c_m x^m$ and $Fx = (\theta x)^2 \varphi_1 x - (\theta_1 x)^2 \varphi_2 x$, the quantities $x_1, x_2, \ldots, x_\mu$ are the roots of $Fx = 0$. We have $F'xdx + \delta Fx = 0$ where

$$\delta Fx = 2\theta x \cdot \varphi_1 x \cdot \delta\theta x - 2\theta_1 x \cdot \varphi_2 x \cdot \delta\theta_1 x.$$

Now the equation $Fx = 0$ implies that $\theta x \cdot \varphi_1 x = \varepsilon\theta_1 x\sqrt{\varphi x}$ and $\theta_1 x \cdot \varphi_2 x = \varepsilon\theta x\sqrt{\varphi x}$ where $\varepsilon = \pm 1$. Thus $F'xdx = 2\varepsilon(\theta x \cdot \delta\theta_1 x - \theta_1 x \cdot \delta\theta x)\sqrt{\varphi x}$ and $\varepsilon \frac{fxdx}{(x-\alpha)\sqrt{\varphi x}} = \frac{2fx(\theta x \cdot \delta\theta_1 x - \theta_1 x \cdot \delta\theta x)}{(x-\alpha)F'x} = \frac{\lambda x}{(x-\alpha)F'x}$ where $\lambda x = (x-\alpha)\lambda_1 x + \lambda\alpha$ and $\lambda_1 x$ are polynomials. This leads to

$$\sum \varepsilon \frac{fx \cdot dx}{(x-\alpha)\sqrt{\varphi x}} = \sum \frac{\lambda_1 x}{F'x} + \lambda\alpha \sum \frac{1}{(x-\alpha)F'x} = -\frac{\lambda\alpha}{F\alpha} + \prod \frac{\lambda x}{(x-\alpha)Fx}$$

(the sums are extended to $x_1, x_2, \ldots, x_\mu$) and then to the relation of the statement. The values of the ε_k are determined by the equations $\theta x_k \sqrt{\varphi_1 x_k} = \varepsilon_k \theta_1 x_k \sqrt{\varphi_2 x_k}$.

In a second theorem, Abel explains that the same statement holds in the case in which some of the roots of Fx are multiple, provided that $\theta x \cdot \varphi_1 x$ and $\theta_1 x \cdot \varphi_2 x$

be relatively prime. The third theorem concerns the case in which $f\alpha = 0$, so that $\psi x = \int \frac{fx \cdot dx}{\sqrt{\varphi x}}$ where fx is a polynomial (written for $\frac{fx}{x-\alpha}$). In this case, the right hand side of the relation reduces to

$$C + \prod \frac{fx}{\sqrt{\varphi x}} \log \frac{\theta x \sqrt{\varphi_1 x} + \theta_1 x \sqrt{\varphi_2 x}}{\theta x \sqrt{\varphi_1 x} - \theta_1 x \sqrt{\varphi_2 x}}.$$

On the contrary (theorem IV), when the degree of $(fx)^2$ is less than the degree of φx, the right hand side reduces to $C - \frac{f\alpha}{\sqrt{\varphi\alpha}} \log \frac{\theta\alpha\sqrt{\varphi_1\alpha}+\theta_1\alpha\sqrt{\varphi_2\alpha}}{\theta\alpha\sqrt{\varphi_1\alpha}-\theta_1\alpha\sqrt{\varphi_2\alpha}}$. Abel deals with the case of the integrals $\psi x = \int \frac{dx}{(x-\alpha)^k\sqrt{\varphi x}}$ by successive differentiations starting from $k = 1$ (theorem V).

The sixth theorem concerns the case in which $\deg(fx)^2 < \deg \varphi x$, that is of integrals of the form $\psi x = \int \frac{(\delta_0+\delta_1 x+\ldots+\delta_{\nu'}x^{\nu'})dx}{\sqrt{\beta_0+\beta_1 x+\ldots+\beta_\nu x^\nu}}$ where $\nu' = \frac{\nu-1}{2} - 1$ when ν is odd and $\nu' = \frac{\nu}{2} - 2$ when ν is even or $\nu' = m - 2$ for $\nu = 2m - 1$ or $2m$. In this case, the right hand side of the relation is a constant.

The general case of $\psi x = \int \frac{r dx}{\sqrt{\varphi x}}$ where r is any rational function of x is reduced to the preceding ones by decomposing r in simple elements (theorem VII). As there are $m+n+2$ indeterminate coefficients $a_0, a_1, \ldots, c_0, c_1, \ldots$, Abel arbitrarily chooses $\mu' = m + n + 1$ quantities $x_1, x_2, \ldots, x_{\mu'}$ and determines $a_0, a_1, \ldots, c_0, c_1, \ldots$ as rational functions of $x_1, x_2, \ldots, x_{\mu'}, \sqrt{\varphi x_1}, \sqrt{\varphi x_2}, \ldots, \sqrt{\varphi x_{\mu'}}$ by the equations $\theta x_k \sqrt{\varphi_1 x_k} = \varepsilon_k \theta_1 x_k \sqrt{\varphi_2 x_k}$, $1 \le k \le \mu'$. Substituting these values in θx and $\theta_1 x$, Fx takes the form $(x-x_1)(x-x_2)\ldots(x-x_{\mu'})R$ where R is a polynomial of degree $\mu-\mu'$ with the roots $x_{\mu'+1}, x_{\mu'+2}, \ldots, x_\mu$. The coefficients of R are rational functions of $x_1, x_2, \ldots, x_{\mu'}, \sqrt{\varphi x_1}, \sqrt{\varphi x_2}, \ldots, \sqrt{\varphi x_{\mu'}}$. Putting $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{\mu_1} = 1$, $\varepsilon_{\mu_1+1} = \varepsilon_{\mu_1+2} = \ldots = \varepsilon_{\mu'} = -1$, $x_{\mu_1+1} = x'_1, x_{\mu_1+2} = x'_2, \ldots, x_{\mu^{\cdot}} = x'_{\mu_2}$ and $x_{\mu'+1} = y_1, x_{\mu'+2} = y_2, \ldots, x_\mu = y_{\nu'}$, Abel rewrites the relation of the statement in the form $\psi x_1 + \psi x_2 + \ldots + \psi x_{\mu_1} - \psi x'_1 - \psi x'_2 - \ldots - \psi x'_{\mu_2} = v - \varepsilon_{\mu'+1}\psi y_1 - \varepsilon_{\mu^{\cdot}+2}\psi y_2 - \ldots - \varepsilon_\mu \psi y_{\nu'}$, where $x_1, x_2, \ldots, x_{\mu_1}, x'_1, x'_2, \ldots, x'_{\mu_2}$ are independent variables and $y_1, y_2, \ldots, y_{\nu'}$ algebraic functions of these variables. He determines the minimum value of $\nu' = \mu - \mu' = \mu - m - n - 1$, where $\mu = \sup(2n + \nu_1, 2m + \nu_2)$, ν_1 and ν_2 denoting the respective degrees of $\varphi_1 x$ and $\varphi_2 x$. The mean value of $2n + \nu_1$ and $2m + \nu_2$ is $m + n + \frac{\nu_1+\nu_2}{2}$. Thus $\nu' \ge \frac{\nu_1+\nu_2}{2} - 1 = \frac{\nu}{2} - 1$ where ν is the degree of φ, and this minimum value, which is the same as that of the theorem VI is attained (theorem VIII). The signs ε_j are determined by the equations $\theta y_j \sqrt{\varphi_1 y_j} = -\varepsilon_j \theta_1 y_j \sqrt{\varphi_2 y_j}$, $1 \le j \le \nu'$. Naturally, when some of the x_k or of the x'_k are equal, one must replace the corresponding equation $\theta x_k \sqrt{\varphi_1 x_k} = \varepsilon_k \theta_1 x_k \sqrt{\varphi_2 x_k}$ by a certain number of derivatives of this equation.

The memoir X (*Œuvres*, t. II, p. 55–66), unpublished by Abel, *Sur la comparaison des transcendantes*, gives a testimony of an early form of Abel theorem. It begins by the same demonstration as in the large memoir for the french Academy, to reach a relation of the form

$$\psi x_1 + \psi x_2 + \ldots + \psi x_\mu = C + \rho - (\psi z_1 + \psi z_2 + \ldots + \psi z_\nu) \qquad (70)$$

where $\psi x = \int f(x, y)dx$, y being an algebraic function of x and $f(x, y)$ a rational function of x and y. Here $z_1, z_2, \ldots, z_\nu$ are algebraic functions of $x_1, x_2, \ldots, x_\mu$, C is a constant and ρ is a function algebraic and logarithmic of $x_1, x_2, \ldots, x_\mu$. Abel proposes a method to compute C in the hypothesis in which $\mu > \nu$; we shall see examples of it later.

In the following, he applies his theorem to the particular case in which y is a rational function of x, defined by the equation $\alpha + \alpha_1 y = 0$ where α, α_1 are polynomials in x. Then the auxiliary equation $\theta y = 0$ is of degree 0 in y, of the form $0 = q = a + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n = s$ and $ydx = \frac{da + xda_1 + x^{n-1}da_{n-1}}{\frac{ds}{dx}} \frac{\alpha}{\alpha_1}$. The n quantities $a, a_1, \ldots, a_{n-1}$ are determined in function of the n independent variables $x_1, x_2, \ldots, x_n$ by writing that these variables are roots of the equation $s = 0$, and $\nu = 0$. When $y = x^m$, $\psi x = \frac{x^{m+1}}{m+1}$ and the theorem states that

$$\frac{1}{m+1}(x_1^{m+1} + x_2^{m+1} + \ldots + x_n^{m+1}) = -\int (P_m da + P_{m+1}da_1 + \ldots + P_{m+n-1}da_{n-1})$$

where $P_k = \frac{x_1^k}{\frac{ds_1}{dx_1}} + \frac{x_2^k}{\frac{ds_2}{dx_2}} + \ldots + \frac{x_n^k}{\frac{ds_n}{dx_n}}$. Now the left hand side is a polynomial $\frac{1}{m+1} Q_{m+1}$ in $a, a_1, \ldots, a_{n-1}$ and we thus have $P_{m+k} = -\frac{1}{m+1}\frac{\partial Q_{m+1}}{\partial a_k}$. In particular $P_k = -\frac{\partial Q_1}{\partial a_k}$ where $Q_1 = -a_{n-1}$ and this gives

$$P_0 = P_1 = \ldots = P_{n-2} = 0, P_{n-1} = 1, \tag{71}$$

identities used several times by Abel. In the same manner, when $y = \frac{1}{(x-\alpha)^m}$,

$$\psi x = -\frac{1}{m-1}\frac{1}{(x-\alpha)^{m-1}}$$

and

$$\frac{1}{m-1}\left(\frac{1}{(x_1-\alpha)^{m-1}} + \frac{1}{(x_2-\alpha)^{m-1}} + \ldots + \frac{1}{(x_n-\alpha)^{m-1}}\right)$$
$$= \int (P_m^{(0)}da + P_m^{(1)}da_1 + \ldots + P_m^{(n-1)}da_{n-1})$$

where $P_m^{(k)} = \frac{x_1^k}{(x_1-\alpha)^m \frac{ds_1}{dx_1}} + \frac{x_2^k}{(x_2-\alpha)^m \frac{ds_2}{dx_2}} + \ldots + \frac{x_n^k}{(x_n-\alpha)^m \frac{ds_n}{dx_n}}$. Thus we have

$$P_m^{(k)} = \frac{1}{m-1}\frac{\partial Q'_{m-1}}{\partial a_k}$$

where

$$Q'_{m-1} = \frac{1}{(x_1-\alpha)^{m-1}} + \frac{1}{(x_2-\alpha)^{m-1}} + \ldots + \frac{1}{(x_n-\alpha)^{m-1}}.$$

When $m = 1$, $\psi x = \log(x - \alpha)$ and the left hand side of (70) is

$$\log(x_1-\alpha)(x_2-\alpha)\ldots(x_n-\alpha) = \log(-1)^n(a + a_1\alpha + \ldots + a_{n-1}\alpha^{n-1} + \alpha^n).$$

Thus $P_1^{(k)} = -\frac{\alpha^k}{a+a_1\alpha+\ldots+a_{n-1}\alpha^{n-1}+\alpha^n}$.

Now supposing that $s = (a + a_1x + \ldots + x_{\mu-1}x^{\mu-1} + x^{\mu})\varphi x - fx$ where

$$\varphi x = \alpha_1 \text{ and } fx = -(\delta + \delta_1 x + \ldots + \delta_{n-1}x^{n-1}),$$

we have $ydx = \frac{\alpha(da+xda_1+\ldots+x^{\mu-1}da_{\mu-1})}{\frac{ds}{dx}}$ and we see that $\rho = 0$ in (70) if $\deg\alpha < \deg\alpha_1$. The quantities $x_1, x_2, \ldots, x_n$ are related by the equations $a + a_1x_k + \ldots + a_{\mu-1}x_k^{\mu-1} + x_k^{\mu} = \frac{fx_k}{\varphi x_k}$, $1 \le k \le n$. Let $x_1', x_2', \ldots, x_n'$ be another set of quantitites and suppose that $\deg\alpha < \deg\alpha_1$; we have $\psi x_1 + \psi x_2 + \ldots + \psi x_n = \psi x_1' + \psi x_2' + \ldots + \psi x_n'$. Now it is possible, by a convenient choice of $\delta, \delta_1, \ldots, \delta_{n-1}$, to impose $\psi x_n' = \psi x_{n-1}' = \ldots = \psi x_{\mu+1}' = 0$. Thus the theorem is written

$$\psi x_1 + \psi x_2 + \ldots + \psi x_n = \psi x_1' + \psi x_2' + \ldots + \psi x_\mu'.$$

For instance, if $\alpha = 1$ and $\alpha_1 = x$, $\psi x = -\log x$ and $s = \delta + ax + a_1x^2 + \ldots + a_{\mu-1}x^{\mu} + x^{\mu+1}$. Thus $\delta = (-1)^{\mu+1}x_1x_2\ldots x_{\mu+1} = (-1)^{\mu+1}x_1'x_2'\ldots x_{\mu+1}'$ and we may impose $x_2' = x_3' = \ldots = x_{\mu+1}' = 1$ to get $x_1' = x_1x_2\ldots x_{\mu+1}$. In this case, the theorem gives $\log x_1 + \log x_2 + \ldots + \log x_{\mu+1} = \log(x_1x_2\ldots x_{\mu+1})$. A second example is given by $\alpha = 1$, $\alpha_1 = 1+x^2$, $\psi x = -\arctan x$. Let x_1, x_2, x_3 be solutions of the equation $0 = \delta + \delta_1 x + (1 + x^2)(a + x)$; we have $\arctan x_1 + \arctan x_2 + \arctan x_3 = C$ constant and $x_1x_2x_3 = -\delta - a$, $x_1+x_2+x_3 = -a$, $x_1x_2+x_1x_3+x_2x_3 = \delta_1 + 1$. Thus $x_1 + x_2 + x_3 - x_1x_2x_3 = \delta$ and $x_1x_2 + x_1x_3 + x_2x_3 - 1 = \delta_1$. Now putting $x_3 = x_2'$, $x_2 = -x_2'$ and $x_1 = x_1'$, we get $C = \arctan x_1'$ and $x_1' + x_1'(x_2')^2 = \delta$, $1 + (x_2')^2 = -\delta_1$, whence $x_1' = -\frac{\delta}{\delta_1} = \frac{x_1+x_2+x_3-x_1x_2x_3}{1-x_1x_2-x_1x_3-x_2x_3}$. Thus the theorem gives $\arctan x_1 + \arctan x_2 + \arctan x_3 = \arctan\frac{x_1+x_2+x_3-x_1x_2x_3}{1-x_1x_2-x_1x_3-x_2x_3}$.

At the end of this memoir, Abel generalises the relations (71). Considering the integral

$$\int fx \cdot dx = \psi x + \sum A\log(x - \delta)$$

where fx and ψx are rational functions and the auxiliary equation $\varphi x = a + a_1x + \ldots + a_nx^n = 0$, with the roots $x_1, x_2, \ldots, x_n$. By the theorem

$$\int fx_1 \cdot dx_1 + \int fx_2 \cdot dx_2 + \ldots + \int fx_n \cdot dx_n$$
$$= \psi x_1 + \psi x_2 + \ldots + \psi x_n + \sum A\log(x_1 - \delta)(x_2 - \delta)\ldots(x_n - \delta) = \rho$$

where $-d\rho = da\left(\frac{fx_1}{\varphi' x_1} + \frac{fx_2}{\varphi' x_2} + \ldots + \frac{fx_n}{\varphi' x_n}\right) + da_1\left(\frac{x_1\cdot fx_1}{\varphi' x_1} + \frac{x_2\cdot fx_2}{\varphi' x_2} + \ldots + \frac{x_n\cdot fx_n}{\varphi' x_n}\right) + \ldots + da_1\left(\frac{x_1^n fx_1}{\varphi' x_1} + \frac{x_2^n fx_2}{\varphi' x_2} + \ldots + \frac{x_n^n fx_n}{\varphi' x_n}\right)$.

Now $\psi x_1 + \psi x_2 + \ldots + \psi x_n$ is a rational function p of $a, a_1, \ldots, a_n$ and $(x_1 - \delta)(x_2 - \delta)\ldots(x_n - \delta) = (-1)^n\frac{\varphi\delta}{a_n}$ so that $\rho = p + \sum A(\log\varphi\delta - \log a_n)$ and

$$\frac{\partial\rho}{\partial a_m} = \frac{\partial p}{\partial a_m} + \sum A\left(\frac{1}{\varphi\delta}\frac{\partial\varphi\delta}{\partial a_m} - \frac{1}{a_n}\frac{\partial a_n}{\partial a_m}\right)$$
$$= -\left(\frac{x_1^m fx_1}{\varphi' x_1} + \frac{x_2^m fx_2}{\varphi' x_2} + \ldots + \frac{x_n^m fx_n}{\varphi' x_n}\right).$$

Abel deduces that $\frac{x_1^m fx_1}{\varphi' x_1} + \frac{x_2^m fx_2}{\varphi' x_2} + \ldots + \frac{x_n^m fx_n}{\varphi' x_n} = -\frac{\partial p}{\partial a_m} - \sum \frac{A\delta^m}{\varphi\delta} + \sum \frac{A}{a_n}\left(\frac{1}{2} \pm \frac{1}{2}\right)$ where the superior sign is taken when $m = n$ and the inferior sign when $m < n$. For $fx = 1$, $\psi x = x$, $p = x_1 + x_2 + \ldots + x_n = -\frac{a_{n-1}}{a_n}$ and $A = 0$. We find back (71) and the relation

$$\frac{x_1^n}{\varphi' x_1} + \frac{x_2^n}{\varphi' x_2} + \ldots + \frac{x_n^n}{\varphi' x_n} = -\frac{a_{n-1}}{a_n^2}.$$

For $fx = \frac{1}{x-\delta}$, $p = 0$ and $A = 1$; if $Fx = \beta + \beta_1 x + \ldots + \beta_n x^n$ we have

$$\frac{Fx_1}{(x_1-\delta)\varphi' x_1} + \frac{Fx_2}{(x_2-\delta)\varphi' x_2} + \ldots + \frac{Fx_n}{(x_n-\delta)\varphi' x_n} = \frac{\beta_n}{a_n} - \frac{F\delta}{\varphi\delta}$$

and other relations by differentiating this one.

6 Elliptic functions

Abel is the founder of the theory of elliptic functions. He partook this glory with Jacobi alone, for Gauss did not publish the important work he had done in this field; the 'grand prix' of the parisian Academy of sciences was awarded to Abel and Jacobi for their work on elliptic functions in 1830, after Abel's death. Abel's work on elliptic functions was published in the second and the third volumes of Crelle's *Journal* (1827–1828), in a large memoir titled *Recherches sur les finctions elliptiques* (*Œuvres*, t. I, p. 263–388).

Abel briefly recalls the main results of Euler, Lagrange and Legendre on elliptic integrals and defines his elliptic function $\varphi\alpha = x$ by the relation

$$\alpha = \int_0 \frac{dx}{\sqrt{(1-c^2x^2)(1+e^2x^2)}} \tag{72}$$

where c and e are real numbers. This definition is equivalent to the differential equation

$$\varphi'\alpha = \sqrt{(1-c^2\varphi^2\alpha)(1+e^2\varphi^2\alpha)}$$

with $\varphi(0) = 0$. Abel puts $f\alpha = \sqrt{1-c^2\varphi^2\alpha}$ and $F\alpha = \sqrt{1+e^2\varphi^2\alpha}$ and explains that the principal aim of his memoir is the resolution of the algebraic equation of degree m^2 which gives $\varphi\alpha$, $f\alpha$, $F\alpha$ when one knows $\varphi(m\alpha)$, $f(m\alpha)$, $F(m\alpha)$ (cf. our §3).

The first paragraph (p. 266–278) of Abel's memoir is devoted to the study of the functions $\varphi\alpha$, $f\alpha$ and $F\alpha$. According to (72), α is a positive increasing function of x for $0 \le x \le \frac{1}{c}$. Thus $\varphi\alpha$ is a positive increasing function of α for $0 \le \alpha \le \frac{\omega}{2}$ $= \int\limits_0^{1/c} \frac{dx}{\sqrt{(1-c^2x^2)(1+e^2x^2)}}$ and we have $\varphi\left(\frac{\omega}{2}\right) = \frac{1}{c}$. Since α is an odd function of x, $\varphi(-\alpha) = -\varphi(\alpha)$. Now Abel puts ix instead of x in (72) (where $i = \sqrt{-1}$) and gets a purely imaginary value $\alpha = i\beta$, so that $xi = \varphi(\beta i)$ where $\beta = \int\limits_0^x \frac{dx}{\sqrt{(1+c^2x^2)(1-e^2x^2)}}$. We see that β is a positive increasing function of x for $0 \le x \le \frac{1}{e}$ and that x is a positive increasing function of β for

$$0 \le \beta \le \frac{\varpi}{2} = \int\limits_0^{1/e} \frac{dx}{\sqrt{(1+c^2x^2)(1-e^2x^2)}}$$

and we have $\varphi\left(\frac{\varpi i}{2}\right) = i\frac{1}{e}$. Abel notes that the exchange of c and e transforms $\frac{\varphi(\alpha i)}{i}$ in $\varphi\alpha$, $f(\alpha i)$ in $F\alpha$, $F(\alpha i)$ in $f\alpha$ and exchanges ω and ϖ.

The function $\varphi\alpha$ is known for $-\frac{\omega}{2} \le \alpha \le \frac{\omega}{2}$ and for $\alpha = \beta i$ with $-\frac{\varpi}{2} \le \beta \le \frac{\varpi}{2}$. Abel extends its definition to the entire complex domain by the addition theorem:

$$\begin{aligned}
\varphi(\alpha+\beta) &= \frac{\varphi\alpha \cdot f\beta \cdot F\beta + \varphi\beta \cdot f\alpha \cdot F\alpha}{1 + e^2c^2\varphi^2\alpha \cdot \varphi^2\beta},\\
f(\alpha+\beta) &= \frac{f\alpha \cdot f\beta - c^2\varphi\alpha \cdot \varphi\beta \cdot F\alpha \cdot F\beta}{1 + e^2c^2\varphi^2\alpha \cdot \varphi^2\beta},\\
F(\alpha+\beta) &= \frac{F\alpha \cdot F\beta + e^2\varphi\alpha \cdot \varphi\beta \cdot f\alpha \cdot f\beta}{1 + e^2c^2\varphi^2\alpha \cdot \varphi^2\beta}.
\end{aligned} \tag{73}$$

This theorem is a consequence of Euler addition theorem for elliptic integrals, but Abel directly proves it by differentiating with respect to α and using $\varphi'\alpha = f\alpha \cdot F\alpha$, $f'\alpha = -c^2\varphi\alpha \cdot F\alpha$, $F'\alpha = e^2\varphi\alpha \cdot f\alpha$. Thus, denoting by r the right hand side of the first formula, he finds that $\frac{\partial r}{\partial \alpha} = \frac{\partial r}{\partial \beta}$ which shows that r is a function of $\alpha+\beta$. As $r = \varphi\alpha$ when $\beta = 0$, this gives $r = \varphi(\alpha+\beta)$. From (73), Abel deduces

$$\begin{aligned}
\varphi(\alpha+\beta)+\varphi(\alpha-\beta) &= \frac{2\varphi\alpha\cdot f\beta\cdot F\beta}{R},\ \varphi(\alpha+\beta)-\varphi(\alpha-\beta) = \frac{2\varphi\beta\cdot f\alpha\cdot F\alpha}{R}, \\
f(\alpha+\beta)+f(\alpha-\beta) &= \frac{2f\alpha\cdot f\beta}{R},\ f(\alpha+\beta)-f(\alpha-\beta) = \frac{-2c^2\varphi\alpha\cdot\varphi\beta\cdot F\alpha\cdot F\beta}{R},\\
F(\alpha+\beta)+F(\alpha-\beta) &= \frac{2F\alpha\cdot F\beta}{R},\ F(\alpha+\beta)-F(\alpha-\beta) = \frac{2e^2\varphi\alpha\cdot\varphi\beta\cdot f\alpha\cdot f\beta}{R}
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
\varphi(\alpha+\beta)\varphi(\alpha-\beta) &= \frac{\varphi^2\alpha - \varphi^2\beta}{R},\ f(\alpha+\beta)f(\alpha-\beta) = \frac{f^2\beta - c^2\varphi^2\alpha\cdot F^2\beta}{R} \\
F(\alpha+\beta)F(\alpha-\beta) &= \frac{F^2\beta + e^2\varphi^2\alpha\cdot f^2\beta}{R}
\end{aligned} \tag{75}$$

where $R = 1 + e^2c^2\varphi^2\alpha\,\varphi^2\beta$.

On the other hand $f\left(\pm\frac{\omega}{2}\right) = F\left(\pm\frac{\varpi}{2}i\right) = 0$ give

$$\varphi\left(\alpha \pm \frac{\omega}{2}\right) = \pm\frac{1}{c}\frac{f\alpha}{F\alpha},\ f\left(\alpha \pm \frac{\omega}{2}\right) = \mp\sqrt{e^2+c^2}\frac{\varphi\alpha}{F\alpha}, \tag{76}$$
$$F\left(\alpha \pm \frac{\omega}{2}\right) = \frac{\sqrt{e^2+c^2}}{c}\frac{1}{F\alpha},\ \varphi\left(\alpha \pm \frac{\varpi}{2}i\right) = \pm\frac{i}{e}\frac{F\alpha}{f\alpha},$$
$$F\left(\alpha \pm \frac{\varpi}{2}i\right) = \pm i\sqrt{e^2+c^2}\frac{\varphi\alpha}{f\alpha},\ f\left(\alpha \pm \frac{\varpi}{2}i\right) = \frac{\sqrt{e^2+c^2}}{e}\frac{1}{f\alpha}.$$

These relations imply that

$$\varphi\left(\frac{\omega}{2}+\alpha\right) = \varphi\left(\frac{\omega}{2}-\alpha\right),\ f\left(\frac{\omega}{2}+\alpha\right) = -f\left(\frac{\omega}{2}-\alpha\right), \tag{77}$$
$$F\left(\frac{\omega}{2}+\alpha\right) = F\left(\frac{\omega}{2}-\alpha\right),\ \varphi\left(\frac{\varpi}{2}i+\alpha\right) = \varphi\left(\frac{\varpi}{2}i-\alpha\right),$$
$$F\left(\frac{\varpi}{2}i+\alpha\right) = -F\left(\frac{\varpi}{2}i-\alpha\right),\ f\left(\frac{\varpi}{2}i+\alpha\right) = f\left(\frac{\varpi}{2}i-\alpha\right)$$

and $\varphi\left(\alpha \pm \frac{\omega}{2}\right)\varphi\left(\alpha + \frac{\varpi}{2}i\right) = \pm\frac{i}{ce}$, $F\left(\alpha \pm \frac{\omega}{2}\right) = F\alpha\frac{\sqrt{e^2+c^2}}{c} = f\left(\alpha \pm \frac{\varpi}{2}i\right) f\alpha$. We deduce that $\varphi\left(\frac{\omega}{2} + \frac{\varpi}{2}i\right) = f\left(\frac{\omega}{2} + \frac{\varpi}{2}i\right) = F\left(\frac{\omega}{2} + \frac{\varpi}{2}i\right) = \frac{1}{0}$ i.e. infinity. From (77) we have

$$\varphi(\alpha+\omega) = -\varphi\alpha = \varphi(\alpha+\varpi i),\ f(\alpha+\omega) = -f\alpha = -f(\alpha+\varpi i), \tag{78}$$
$$F(\alpha+\omega) = F\alpha = -F(\alpha+\varpi i)$$

and

$$\varphi(2\omega+\alpha) = \varphi\alpha = \varphi(2\varpi i+\alpha) = \varphi(\omega+\varpi i+\alpha), \tag{79}$$
$$f(2\omega+\alpha) = f\alpha = f(\varpi i+\alpha),\ F(\omega+\alpha) = F\alpha = F(2\varpi i+\alpha).$$

Thus the functions $\varphi\alpha$, $f\alpha$, $F\alpha$ are *periodic*:

$$\varphi(m\omega+n\varpi i \pm \alpha) = \pm(-1)^{m+n}\varphi\alpha,\ f(m\omega+n\varpi i \pm \alpha) = (-1)^m f\alpha, \tag{80}$$
$$F(m\omega+n\varpi i \pm \alpha) = (-1)^n F\alpha.$$

The equation $\varphi(\alpha+\beta i) = 0$ is equivalent to $\frac{\varphi\alpha\cdot f(\beta i)F(\beta i)+\varphi(\beta i)f\alpha\cdot F\alpha}{1+e^2c^2\varphi^2\alpha\cdot\varphi^2(\beta i)} = 0$ (cf. (73)) and, as $\varphi\alpha$, $f(\beta i)$, $F(\beta i)$ are real and $\varphi(\beta i)$ is purely imaginary, this signifies $\varphi\alpha\cdot f(\beta i)F(\beta i) = 0$ and $\varphi(\beta i)f\alpha\cdot F\alpha = 0$. These equations are satisfied by $\varphi\alpha = \varphi(\beta i) = 0$ or by $f(\beta i)F(\beta i) = f\alpha\cdot F\alpha = 0$. The first solution gives $\alpha = m\omega$, $\beta = n\varpi$ and it fits, for $\varphi(m\omega+n\varpi i) = 0$. The second solution gives $\alpha = \left(m+\frac{1}{2}\right)\omega$, $\beta = \left(n+\frac{1}{2}\right)\varpi$ and it does not fit, for $\varphi\left(\left(m+\frac{1}{2}\right)\omega + \left(n+\frac{1}{2}\right)\varpi i\right) = \frac{1}{0}$. In the same way, Abel determines the roots of the equation $fx = 0$, which are $x = \left(m+\frac{1}{2}\right)\omega + n\varpi i$ and those of the equation $Fx = 0$, which are $x = m\omega + \left(n+\frac{1}{2}\right)\varpi i$. From these results and the formulae

$$\varphi x = \frac{i}{ec}\frac{1}{\varphi\left(x-\frac{\omega}{2}-\frac{\varpi}{2}i\right)},\ fx = \frac{\sqrt{e^2+c^2}}{e}\frac{1}{f\left(x-\frac{\varpi}{2}i\right)},\ Fx = \frac{\sqrt{e^2+c^2}}{c}\frac{1}{F\left(x-\frac{\omega}{2}\right)}, \tag{81}$$

he deduces the poles of the functions φx, fx, Fx, which are $x = \left(m+\frac{1}{2}\right)\omega + \left(n+\frac{1}{2}\right)\varpi i$.

From (74) $\varphi x - \varphi a = \frac{2\varphi\left(\frac{x-a}{2}\right)f\left(\frac{x+a}{2}\right)F\left(\frac{x+a}{2}\right)}{1+e^2c^2\varphi^2\left(\frac{x+a}{2}\right)\varphi^2\left(\frac{x-a}{2}\right)}$. Thus the equation $\varphi x = \varphi a$ is equivalent to $\varphi\left(\frac{x-a}{2}\right) = 0$ or $f\left(\frac{x+a}{2}\right) = 0$ or $F\left(\frac{x+a}{2}\right) = 0$ or $\varphi\left(\frac{x-a}{2}\right) = \frac{1}{0}$ or $\varphi\left(\frac{x+a}{2}\right) = 0$. Thus the solutions are $x = (-1)^{m+n}a + m\omega + n\varpi i$. In the same way, the solutions of $fx = fa$ are given by $x = \pm a + 2m\omega + n\varpi i$ and those of $Fx = Fa$ by $x = \pm a + m\omega + 2n\varpi i$.

The second paragraph (p. 279–281) of Abel's memoir contains the proof by complete induction that $\varphi(n\beta)$, $f(n\beta)$ and $F(n\beta)$ are rational functions of $\varphi\beta$, $f\beta$ and $F\beta$ when n is an integer. Writing $\varphi(n\beta) = \frac{P_n}{Q_n}$, $f(n\beta) = \frac{P'_n}{Q_n}$ and $F(n\beta) = \frac{P''_n}{Q_n}$ where P_n, P'_n, P''_n and Q_n are polynomials in $\varphi\beta$, $f\beta$ and $F\beta$, we have, by (74)

$$\frac{P_{n+1}}{Q_{n+1}} = -\frac{P_{n-1}}{Q_{n-1}} + \frac{2f\beta\cdot F\beta\frac{P_n}{Q_n}}{1+e^2c^2\varphi^2\beta\frac{P_n^2}{Q_n^2}} = \frac{-P_{n-1}(Q_n^2+c^2e^2x^2P_n^2)+2P_nQ_nQ_{n-1}yz}{Q_{n-1}R_n}$$

where $x = \varphi\beta$, $y = f\beta$, $z = F\beta$ and $R_n = Q_n^2 + e^2c^2x^2P_n^2$, and we conclude that

$$Q_{n+1} = Q_{n-1}R_n,\ P_{n+1} = -P_{n-1}R_n + 2yzP_nq_nQ_{n-1}.$$

In the same way $P'_{n+1} = -P'_{n-1}R_n + 2yP'_nQ_nQ_{n-1}$ and $P''_{n+1} = -P''_{n-1}R_n + 2yP''_nQ_nQ_{n-1}$. These recursion formulae, together with $y^2 = 1 - c^2x^2$ and $z^2 = 1 + e^2x^2$, show that Q_n, $\frac{P_{2n}}{xyz}$, $\frac{P_{2n+1}}{x}$, P'_{2n}, $\frac{P'_{2n+1}}{y}$, P''_{2n} and $\frac{P''_{2n+1}}{z}$ are polynomials in x^2.

The equations $\varphi(n\beta) = \frac{P_n}{Q_n}$, $f(n\beta) = \frac{P'_n}{Q_n}$ and $F(n\beta) = \frac{P''_n}{Q_n}$ are studied in paragraph III (p. 282–291). When n is even, here noted $2n$, the first equation is written

$$\varphi(2n\beta) = xyz\psi(x^2) = x\psi(x^2)\sqrt{(1-c^2x^2)(1+e^2x^2)}$$

or $\varphi^2(2n\beta) = x^2(\psi x^2)^2(1-c^2x^2)(1+e^2x^2) = \theta(x^2)$, where $x = \varphi\beta$ is one of the roots. If $x = \varphi\alpha$ is another root, $\varphi(2n\alpha) = \pm\varphi(2n\beta)$ and, by the preceding properties,

$$\alpha = \pm((-1)^{m+\mu}2n\beta + m\omega + \mu\varpi i)).$$

Thus the roots of our equation are $\varphi\alpha = \pm\varphi\left((-1)^{m+\mu}\beta + \frac{m}{2n}\omega + \frac{\mu}{2n}\varpi i\right)$, formula in which we may replace m and μ by the remainders of their division by $2n$, because of (80). Abel remarks that, when $0 \le m, \mu < 2n$, all the values of $\varphi\alpha$ so obtained are different. It results that the total number of roots is equal to $8n^2$ and this is the degree of the equation, for it cannot have any multiple root. When $n = 1$, the equation is $(1+e^2c^2x^4)\varphi^2(2\beta) = 4x^2(1-c^2x^2)(1+e^2x^2)$ and its roots are $\pm\varphi\beta$, $\pm\varphi\left(-\beta+\frac{\omega}{2}\right)$, $\pm\varphi\left(-\beta+\frac{\varpi}{2}i\right)$ and $\pm\varphi\left(\beta+\frac{\omega}{2}+\frac{\varpi}{2}i\right)$.

When n is an odd number, here written $2n+1$, the equation is $\varphi(2n+1)\beta = \frac{P_{2n+1}}{Q_{2n+1}}$ and its roots $x = \varphi\left((-1)^{m+\mu}\beta + \frac{m}{2n+1}\omega + \frac{\mu}{2n+1}\varpi i\right)$ where $-n \le m,\ \mu \le n$. The number of these roots is $(2n+1)^2$ and it is the degree of the equation. For example $n = 1$ gives an equation of degree 9 with the roots $\varphi\beta$, $\varphi\left(-\beta - \frac{\omega}{3}\right)$, $\varphi\left(-\beta + \frac{\omega}{3}\right)$, $\varphi\left(-\beta - \frac{\varpi}{3}i\right)$, $\varphi\left(-\beta + \frac{\varpi}{3}i\right)$, $\varphi\left(\beta - \frac{\omega}{3} - \frac{\varpi}{3}i\right)$, $\varphi\left(\beta - \frac{\omega}{3} + \frac{\varpi}{3}i\right)$, $\varphi\left(\beta + \frac{\omega}{3} - \frac{\varpi}{3}i\right)$ and $\varphi\left(\beta + \frac{\omega}{3} + \frac{\varpi}{3}i\right)$.

Abel studies in the same way the equations $f(n\beta) = \frac{P'_n}{Q_n}$ and $F(n\beta) = \frac{P''_n}{Q_n}$ of which the roots are respectively $y = f\left(\beta + \frac{2m}{n}\omega + \frac{\mu}{n}\varpi i\right)$ and $z = F\left(\beta + \frac{m}{n}\omega + \frac{2\mu}{n}\varpi i\right)$, $(0 \le m,\ \mu < n$). Each of these equations is of degree n^2.

There are particular cases: $P^2_{2n} = 0$, with the roots $x = \pm\varphi\left(\frac{m}{2n}\omega + \frac{\mu}{2n}\varpi i\right)$ $(0 \le m, \mu \le 2n-1)$, $P_{2n+1} = 0$, with the roots $x = \varphi\left(\frac{m}{2n+1}\omega + \frac{\mu}{2n+1}\varpi i\right)$ $(-n \le m, \mu \le n)$, $P'_n = 0$, with the roots $y = f\left(\left(2m + \frac{1}{2}\right)\frac{\omega}{n} + \frac{\mu}{n}\varpi i\right)$, $P''_n = 0$, with the roots $z = F\left(\frac{m}{n}\omega + \left(2\mu + \frac{1}{2}\right)\frac{\varpi i}{n}\right)$ $(0 \le m, \mu \le n-1)$ and $Q_{2n} = 0$, with the roots $x = \varphi\left(\left(m + \frac{1}{2}\right)\frac{\omega}{2n} + \left(\mu + \frac{1}{2}\right)\frac{\varpi i}{2n}\right)$ $(0 \le m, \mu \le 2n-1)$, $Q_{2n+1} = 0$ with the roots $x = (-1)^{m+\mu}\varphi\left(\left(m + \frac{1}{2}\right)\frac{\omega}{2n+1} + \left(\mu + \frac{1}{2}\right)\frac{\varpi i}{2n+1}\right)$ $(-n \le m, \mu \le n, (m,\mu) \le (n,n))$.

The algebraic solution of the equations $\varphi(n\beta) = \frac{P_n}{Q_n}$, $f(n\beta) = \frac{P'_n}{Q_n}$ and $F(n\beta) = \frac{P''_n}{Q_n}$ is given in paragraph IV (p. 291–305). It is sufficient to deal with the case in which n is a prime number. The case $n = 2$ is easy for if $x = \varphi\frac{\alpha}{2}$, $y = f\frac{\alpha}{2}$ and $z = F\frac{\alpha}{2}$, we have

$$f\alpha = \frac{y^2 - c^2x^2z^2}{1 + e^2c^2x^4} = \frac{1 - 2c^2x^2 - c^2e^2x^4}{1 + e^2c^2x^4},$$
$$F\alpha = \frac{z^2 + e^2y^2x^2}{1 + e^2c^2x^4} = \frac{1 + 2e^2x^2 - e^2c^2x^4}{1 + e^2c^2x^4}.$$

Hence $\frac{F\alpha - 1}{1 + f\alpha} = e^2x^2$, $\frac{1 - f\alpha}{F\alpha + 1} = c^2x^2$ and $z^2 = \frac{F\alpha + f\alpha}{1 + f\alpha}$, $y^2 = \frac{F\alpha + f\alpha}{1 + F\alpha}$ and we draw $\varphi\frac{\alpha}{2} = \frac{1}{c}\sqrt{\frac{1 - f\alpha}{1 + F\alpha}} = \frac{1}{e}\sqrt{\frac{F\alpha - 1}{f\alpha + 1}}$, $f\frac{\alpha}{2} = \sqrt{\frac{F\alpha + f\alpha}{1 + F\alpha}}$, $F\frac{\alpha}{2} = \sqrt{\frac{F\alpha + f\alpha}{1 + f\alpha}}$. From these formulae, it is possible to express $\varphi\frac{\alpha}{2^n}$, $f\frac{\alpha}{2^n}$, $F\frac{\alpha}{2^n}$ with square roots in function of $\varphi\alpha$, $f\alpha$, $F\alpha$. Taking $\alpha = \frac{\omega}{2}$ as an example, Abel finds

$$\varphi\frac{\omega}{4} = \frac{1}{\sqrt{c^2 + c\sqrt{e^2 + c^2}}} = \frac{\sqrt{c\sqrt{e^2 + c^2} - c^2}}{ec},$$
$$f\frac{\omega}{4} = \frac{1}{e}\sqrt{e^2 + c^2 - c\sqrt{e^2 + c^2}},\ F\frac{\omega}{4} = \sqrt[4]{1 + \frac{e^2}{c^2}} = \sqrt{F\frac{\omega}{2}}.$$

The case n odd was explained in our §3. The essential point was that the auxiliary functions such as $\varphi_1\beta$ are *rational* functions of $\varphi\beta$ because of the addition theorem (73). At the same place, we have dealt with the equation $P_{2n+1} = 0$ (§V of Abel's memoir, p. 305–314)) which determines the quantities $x = \varphi\left(\frac{m\omega + \mu\varpi i}{2n+1}\right)$. We saw that the equation in $r = x^2$ is of degree $2n(n+1)$ and that it may be decomposed in $2n+2$ equations of degree n of which the coefficients are rational functions of the roots of an equation of degree $2n+2$. The equations of degree n are all solvable by radicals, but the equation of degree $2n+2$ is not solvable in general.

In paragraph VI (p. 315–323), Abel gives explicit formulae for

$$\varphi((2n+1)\beta),\ f((2n+1)\beta) \text{ and } F((2n+1)\beta)$$

in function of the quantities $\varphi\left(\beta + \frac{m\omega+\mu\varpi i}{2n+1}\right)$, $f\left(\beta + \frac{m\omega+\mu\varpi i}{2n+1}\right)$, $F\left(\beta + \frac{m\omega+\mu\varpi i}{2n+1}\right)$. Let $P_{2n+1} = Ax^{(2n+1)^2} + \ldots + Bx$, $P'_{2n+1} = A'y^{(2n+1)^2} + \ldots + B'y$, $P''_{2n+1} = A''z^{(2n+1)^2} + \ldots + B''z$ and $Q_{2n+1} = Cx^{(2n+1)^2-1} + \ldots + D = C'y^{(2n+1)^2-1} + \ldots + D' = C''z^{(2n+1)^2-1} + \ldots + D''$ (an even function). From the equations

$$\begin{aligned} Ax^{(2n+1)^2} + \ldots + Bx &= \varphi((2n+1)\beta \cdot (Cx^{(2n+1)^2-1} + \ldots + D), \\ A'y^{(2n+1)^2} + \ldots + B'y &= f((2n+1)\beta \cdot (C'y^{(2n+1)^2-1} + \ldots + D'), \\ A''z^{(2n+1)^2} + \ldots + B''z &= F((2n+1)\beta \cdot (C''z^{(2n+1)^2-1} + \ldots + D''), \end{aligned}$$

considering the sum and the product of the roots, Abel deduces that

$$\begin{aligned} \varphi((2n+1)\beta) &= \frac{A}{C} \sum_{m=-n}^{n} \sum_{\mu=-n}^{n} (-1)^{m+\mu} \varphi\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right) \\ &= \frac{A}{D} \prod_{m=-n}^{n} \prod_{\mu=-n}^{n} \varphi\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right). \end{aligned} \tag{82}$$

In the same way

$$\begin{aligned} f((2n+1)\beta &= \frac{A'}{C'} \sum_{m=-n}^{n} \sum_{\mu=-n}^{n} (-1)^{m} f\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right) \\ &= \frac{A'}{D'} \prod_{m=-n}^{n} \prod_{\mu=-n}^{n} f\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right) \end{aligned} \tag{82'}$$

and

$$\begin{aligned} F((2n+1)\beta) &= \frac{A''}{C''} \sum_{m=-n}^{n} \sum_{\mu=-n}^{n} (-1)^{\mu} F\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right) \\ &= \frac{A''}{D''} \prod_{m=-n}^{n} \prod_{\mu=-n}^{n} F\left(\beta + \frac{m\omega + \mu\varpi i}{2n+1}\right). \end{aligned} \tag{82''}$$

The coefficients $\frac{A}{C}$, $\frac{A'}{C'}$, $\frac{A''}{C''}$, which do not depend on β, are determined by letting β tend towards the pole $\frac{\omega}{2} + \frac{\varpi}{2}i$, for they are the respective limit values of

$$\frac{\varphi((2n+1)\beta)}{\varphi\beta},\ \frac{f((2n+1)\beta)}{f\beta},\ \frac{F((2n+1)\beta)}{F\beta}.$$

Putting $\beta = \frac{\omega}{2} + \frac{\varpi}{2}i + \alpha$, where α tends towards 0, and using (80) and (81), Abel determines $\frac{A}{C} = \frac{1}{2n+1}$, $\frac{A'}{C'} = \frac{A''}{C''} = \frac{(-1)^n}{2n+1}$. Since the limit of $\frac{\varphi((2n+1)\beta)}{\varphi\beta}$ when β tends towards 0 is $2n+1$ we find

$$2n+1 = \frac{A}{D}\prod_{m=1}^{n}\varphi^2\left(\frac{m\omega}{2n+1}\right)\prod_{\mu=1}^{n}\varphi^2\left(\frac{\mu\varpi i}{2n+1}\right) \times \prod_{m=1}^{n}\prod_{\mu=1}^{n}\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right)\varphi^2\left(\frac{m\omega-\mu\varpi i}{2n+1}\right).$$

In the same way, letting β tend respectively towards $\frac{\omega}{2}$ and $\frac{\varpi i}{2}$ we get

$$\begin{aligned}(-1)^n(2n+1) &= \frac{A'}{D'}\prod_{m=1}^{n}f^2\left(\frac{\omega}{2}+\frac{m\omega}{2n+1}\right)\prod_{\mu=1}^{n}f^2\left(\frac{\omega}{2}+\frac{\mu\varpi i}{2n+1}\right)\\ &\quad\times\prod_{m=1}^{n}\prod_{\mu=1}^{n}f^2\left(\frac{\omega}{2}+\frac{m\omega+\mu\varpi i}{2n+1}\right)f^2\left(\frac{\omega}{2}+\frac{m\omega-\mu\varpi i}{2n+1}\right)\\ &= \frac{A''}{D''}\prod_{m=1}^{n}F^2\left(\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)\prod_{\mu=1}^{n}F^2\left(\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)\\ &\quad\times\prod_{m=1}^{n}\prod_{\mu=1}^{n}F^2\left(\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)F^2\left(\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)\end{aligned}$$

from which it is possible to draw the values of $\frac{A}{D}$, $\frac{A'}{D'}$ and $\frac{A''}{D''}$. Abel further simplifies the expressions of $\varphi((2n+1)\beta)$, $f((2n+1)\beta)$ and $F((2n+1)\beta)$ as products by the formulae

$$\frac{\varphi(\beta+\alpha)\varphi(\beta-\alpha)}{\varphi^2\alpha} = -\frac{1-\frac{\varphi^2\beta}{\varphi^2\alpha}}{1-\frac{\varphi^2\beta}{\varphi^2(\alpha+\frac{\omega}{2}+\frac{\varpi}{2}i)}},$$

$$\frac{f(\beta+\alpha)f(\beta-\alpha)}{f^2\left(\frac{\omega}{2}+\alpha\right)} = -\frac{1-\frac{f^2\beta}{f^2(\frac{\omega}{2}+\alpha)}}{1-\frac{f^2\beta}{f^2(\alpha+\frac{\omega}{2}+\frac{\varpi}{2}i)}},$$

$$\frac{F(\beta+\alpha)F(\beta-\alpha)}{F^2\left(\frac{\varpi}{2}i+\alpha\right)} = -\frac{1-\frac{F^2\beta}{F^2(\frac{\varpi}{2}i+\alpha)}}{1-\frac{F^2\beta}{F^2(\alpha+\frac{\omega}{2}+\frac{\varpi}{2}i)}} \text{ (cf. (75) and (81))}$$

and he thus obtains

$$\begin{aligned}&\varphi((2n+1)\beta)\\ &= (2n+1)\varphi\beta\prod_{m=1}^{n}\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{m\omega}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}\prod_{\mu=1}^{n}\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}}\\ &\quad\times\prod_{m=1}^{n}\prod_{\mu=1}^{n}\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{m\omega+\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}\frac{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{m\omega-\mu\varpi i}{2n+1}\right)}}{1-\frac{\varphi^2\beta}{\varphi^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}},\end{aligned} \tag{83}$$

$$f((2n+1)\beta)$$

$$=(-1)^n(2n+1)f\beta\prod_{m=1}^{n}\frac{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{m\omega}{2n+1}\right)}}{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}\prod_{\mu=1}^{n}\frac{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{\mu\varpi i}{2n+1}\right)}}{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}}$$

$$\times\prod_{m=1}^{n}\prod_{\mu=1}^{n}\frac{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}\frac{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}{1-\frac{f^2\beta}{f^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}} \tag{83'}$$

$$F((2n+1)\beta)$$

$$=(-1)^n(2n+1)F\beta\prod_{m=1}^{n}\frac{1-\frac{F^2\beta}{F^2\left(\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}{1-\frac{F^2\beta}{F^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega}{2n+1}\right)}}\prod_{\mu=1}^{n}\frac{1-\frac{F^2\beta}{F^2\left(\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}}{1-\frac{F^2\beta}{F^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{\mu\varpi i}{2n+1}\right)}}$$

$$\times\prod_{m=1}^{n}\prod_{\mu=1}^{n}\frac{1-\frac{F^2\beta}{F^2\left(\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}{1-\frac{F^2\beta}{F^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega+\mu\varpi i}{2n+1}\right)}}\frac{1-\frac{F^2\beta}{F^2\left(\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}}{1-\frac{F^2\beta}{F^2\left(\frac{\omega}{2}+\frac{\varpi}{2}i+\frac{m\omega-\mu\varpi i}{2n+1}\right)}} \tag{83''}$$

expressions of $\varphi((2n+1)\beta)$, $f((2n+1)\beta)$ and $F((2n+1)\beta)$ in rational functions of $\varphi\beta$, $f\beta$ and $F\beta$ respectively. Abel also transforms the last two to have $\frac{f((2n+1)\beta)}{f\beta}$ and $\frac{F((2n+1)\beta)}{F\beta}$ in rational functions of $\varphi\beta$.

In his paragraph VII (p. 323–351), Abel keeps $\alpha = (2n+1)\beta$ fixed in the formulae (82) and (83) and let n tend towards infinity in order to obtain expansions of his elliptic functions in infinite series and infinite products. From (82) with the help of (81), we have

$$\begin{aligned}\varphi\alpha &= \frac{1}{2n+1}\varphi\frac{\alpha}{2n+1}\\ &+\frac{1}{2n+1}\sum_{m=1}^{n}(-1)^m\left(\varphi\left(\frac{\alpha+m\omega}{2n+1}\right)+\varphi\left(\frac{\alpha-m\omega}{2n+1}\right)\right)\\ &+\frac{1}{2n+1}\sum_{\mu=1}^{n}(-1)^\mu\left(\varphi\left(\frac{\alpha+\mu\varpi i}{2n+1}\right)+\varphi\left(\frac{\alpha-\mu\varpi i}{2n+1}\right)\right)\\ &-\frac{i}{ec}\sum_{m=1}^{n}\sum_{\mu=1}^{n}(-1)^{m+\mu}\psi(n-m,n-\mu)\\ &+\frac{i}{ec}\sum_{m=1}^{n}\sum_{\mu=1}^{n}(-1)^{m+\mu}\psi_1(n-m,n-\mu)\end{aligned}$$

where

$$\psi(m,\mu)=\frac{1}{2n+1}\left(\frac{1}{\varphi\left(\frac{\alpha+\left(m+\frac{1}{2}\right)\omega+\left(\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right)}+\frac{1}{\varphi\left(\frac{\alpha-\left(m+\frac{1}{2}\right)\omega-\left(\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right)}\right)$$

and

$$\psi_1(m,\mu)=\frac{1}{2n+1}\left(\frac{1}{\varphi\left(\frac{\alpha+\left(m+\frac{1}{2}\right)\omega-\left(\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right)}+\frac{1}{\varphi\left(\frac{\alpha-\left(m+\frac{1}{2}\right)\omega+\left(\mu+\frac{1}{2}\right)\varpi i}{2n+1}\right)}\right).$$

Now

$$\begin{aligned}A_m&=(2n+1)\left(\varphi\left(\frac{\alpha+m\omega}{2n+1}\right)+\varphi\left(\frac{\alpha-m\omega}{2n+1}\right)\right)\\&=(2n+1)\frac{2\varphi\left(\frac{\alpha}{2n+1}\right)f\left(\frac{m\omega}{2n+1}\right)F\left(\frac{m\omega}{2n+1}\right)}{1+e^2c^2\varphi^2\left(\frac{m\omega}{2n+1}\right)\varphi^2\left(\frac{\alpha}{2n+1}\right)}\end{aligned}$$

and

$$\begin{aligned}B_\mu&=(2n+1)\left(\varphi\left(\frac{\alpha+\mu\varpi i}{2n+1}\right)+\varphi\left(\frac{\alpha-\mu\varpi i}{2n+1}\right)\right)\\&=(2n+1)\frac{2\varphi\left(\frac{\alpha}{2n+1}\right)f\left(\frac{\mu\varpi i}{2n+1}\right)F\left(\frac{\mu\varpi i}{2n+1}\right)}{1+e^2c^2\varphi^2\left(\frac{\mu\varpi i}{2n+1}\right)\varphi^2\left(\frac{\alpha}{2n+1}\right)}\end{aligned}$$

remain bounded and the first part $\frac{1}{2n+1}\varphi\frac{\alpha}{2n+1}+\frac{1}{(2n+1)^2}\sum_{m=1}^{n}(-1)^m(A_m+B_m)$ of $\varphi\alpha$ has 0 for limit when n tends towards ∞. Thus

$$\begin{aligned}\varphi\alpha=&-\frac{i}{ec}\lim\sum_{m=1}^{n}\sum_{\mu=1}^{n}(-1)^{m+\mu}\psi(n-m,n-\mu)\\&+\frac{i}{ec}\lim\sum_{m=1}^{n}\sum_{\mu=1}^{n}(-1)^{m+\mu}\psi_1(n-m,n-\mu).\end{aligned}$$

It remains to compute the limit of $\sum_{m=0}^{n-1}\sum_{\mu=0}^{n-1}(-1)^{m+\mu}\psi(m,\mu)$ for the second part will be deduced from the first by changing the sign of i. We have $\psi(m,\mu)=\frac{1}{2n+1}\frac{2\varphi\left(\frac{\alpha}{2n+1}\right)\theta\left(\frac{\varepsilon_\mu}{2n+1}\right)}{\varphi^2\left(\frac{\alpha}{2n+1}\right)-\varphi^2\left(\frac{\varepsilon_\mu}{2n+1}\right)}$ where $\theta\varepsilon=f\varepsilon F\varepsilon$ and $\varepsilon_\mu=\left(m+\frac{1}{2}\right)\omega+\left(\mu+\frac{1}{2}\right)\varpi i$ (cf. (74) and (75)) and this has for limit $\theta(m,\mu)=\frac{2\alpha}{\alpha^2-\left(\left(m+\frac{1}{2}\right)\omega+\left(\mu+\frac{1}{2}\right)\varpi i\right)^2}$ when n

tends towards ∞. Abel tries to prove that $\sum_{\mu=1}^{n-1}(-1)^\mu\psi(m,\mu)-\sum_{\mu=1}^{n-1}(-1)^\mu\theta(m,\mu)$ is negligible with respect to $\frac{1}{2n+1}$ by estimating the difference $\psi(m,\mu)-\theta(m,\mu)$, but his reasoning is not clear. Then he replaces $\sum_{\mu=1}^{n-1}(-1)^\mu\theta(m,\mu)$ by the sum up to infinity using a sum formula to estimate $\sum_{\mu=n}^{\infty}(-1)^\mu\theta(m,\mu)$, again negligible with respect to $\frac{1}{2n+1}$. He finally obtains

$$\begin{aligned}\varphi\alpha &= \frac{i}{ec}\sum_{m=1}^{\infty}(-1)^m\sum_{\mu=1}^{\infty}(-1)^\mu\left(\frac{2\alpha}{\alpha^2-\left(\left(m+\frac12\right)\omega-\left(\mu+\frac12\right)\varpi i\right)^2}\right.\\ &\qquad\left.-\frac{2\alpha}{\alpha^2-\left(\left(m+\frac12\right)\omega+\left(\mu+\frac12\right)\varpi i\right)^2}\right)\\ &= \frac{1}{ec}\sum_{m=1}^{\infty}(-1)^m\sum_{\mu=1}^{\infty}(-1)^\mu\left(\frac{(2\mu+1)\varpi}{\left(\alpha-\left(m+\frac12\right)\omega\right)^2+\left(\mu+\frac12\right)^2\varpi^2}\right.\\ &\qquad\left.-\frac{(2\mu+1)\varpi}{\left(\alpha+\left(m+\frac12\right)\omega\right)^2+\left(\mu+\frac12\right)^2\varpi^2}\right).\end{aligned}\tag{84}$$

By the same method, Abel obtains

$$\begin{aligned}f\alpha &= \frac{1}{e}\sum_{\mu=0}^{\infty}\left(\sum_{m=0}^{\infty}(-1)^m\frac{2\left(\alpha+\left(m+\frac12\right)\omega\right)}{\left(\alpha+\left(m+\frac12\right)\omega\right)^2+\left(\mu+\frac12\right)^2\varpi^2}\right.\\ &\qquad\left.-\sum_{m=0}^{\infty}(-1)^m\frac{2\left(\alpha-\left(m+\frac12\right)\omega\right)}{\left(\alpha-\left(m+\frac12\right)\omega\right)^2+\left(\mu+\frac12\right)^2\varpi^2}\right),\end{aligned}\tag{84'}$$

$$\begin{aligned}F\alpha &= \frac{1}{c}\sum_{m=0}^{\infty}\left(\sum_{\mu=0}^{\infty}(-1)^\mu\frac{(2\mu+1)\varpi}{\left(\alpha-\left(m+\frac12\right)\omega\right)^2+\left(\mu+\frac12\right)^2\varpi^2}\right.\\ &\qquad\left.+\sum_{\mu=0}^{\infty}(-1)^\mu\frac{(2\mu+1)\varpi}{\left(\alpha+\left(m+\frac12\right)\omega\right)^2+\left(\mu+\frac12\right)^2\varpi^2}\right).\end{aligned}\tag{84''}$$

He deals with the formulae (83) in the same way by taking the logarithms. For any constants k and ℓ, $\dfrac{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+k}{2n+1}\right)}}{1-\frac{\varphi^2\left(\frac{\alpha}{2n+1}\right)}{\varphi^2\left(\frac{m\omega+\mu\varpi i+\ell}{2n+1}\right)}}$ has a limit equal to $\dfrac{1-\frac{\alpha^2}{(m\omega+\mu\varpi i+k)^2}}{1-\frac{\alpha^2}{(m\omega+\mu\varpi i+\ell)^2}}$. Abel tries to proves that the difference of the logarithms $\psi(m,\mu)$ and $\theta(m,\mu)$ of these expressions is dominated by $\frac{1}{(2n+1)^2}$, with the difficulty that m and μ vary in the sum to be computed. He deduces that the difference $\sum_{\mu=1}^{n}\psi(m,\mu)-\sum_{\mu=1}^{n}\theta(m,\mu)$ is negligible with respect to $\frac{1}{2n+1}$ and replaces $\sum_{\mu=1}^{n}\theta(m,\mu)$ by $\sum_{\mu=1}^{\infty}\theta(m,\mu)$. The proof that $\sum_{\mu=n+1}^{\infty}\theta(m,\mu)=\sum_{\mu=1}^{\infty}\theta(m,\mu+n)$ is negligible with respect to $\frac{1}{2n+1}$ is based on the expansion of $\theta(m,\mu+n)$ in powers of α but it is not sufficient. Abel finally gets $\lim\sum_{m=1}^{n}\sum_{\mu=1}^{n}\psi(m,\mu)=\sum_{m=1}^{\infty}\sum_{\mu=1}^{\infty}\theta(m,\mu)$. He deals in the same way with the simple products in (83) and obtains

$$\varphi\alpha=\alpha\prod_{m=1}^{\infty}\left(1-\frac{\alpha^2}{(m\omega)^2}\right)\prod_{\mu=1}^{\infty}\left(1+\frac{\alpha^2}{(\mu\varpi)^2}\right)$$

$$\times\prod_{m=1}^{\infty}\left(\prod_{\mu=1}^{\infty}\frac{1-\frac{\alpha^2}{(m\omega+\mu\varpi i)^2}}{1-\frac{\alpha^2}{\left(\left(m-\frac{1}{2}\right)\omega+\left(\mu-\frac{1}{2}\right)\varpi i\right)^2}}\prod_{\mu=1}^{\infty}\frac{1-\frac{\alpha^2}{(m\omega-\mu\varpi i)^2}}{1-\frac{\alpha^2}{\left(\left(m-\frac{1}{2}\right)\omega-\left(\mu-\frac{1}{2}\right)\varpi i\right)^2}}\right),$$

$$f\alpha=\prod_{m=1}^{\infty}\left(1-\frac{\alpha^2}{\left(m-\frac{1}{2}\right)^2\omega^2}\right)$$

$$\times\prod_{m=1}^{\infty}\prod_{\mu=1}^{\infty}\frac{1-\frac{\alpha^2}{\left(\left(m-\frac{1}{2}\right)\omega+\mu\varpi i\right)^2}}{1-\frac{\alpha^2}{\left(\left(m-\frac{1}{2}\right)\omega+\left(\mu-\frac{1}{2}\right)\varpi i\right)^2}}\,\frac{1-\frac{\alpha^2}{\left(\left(m-\frac{1}{2}\right)\omega-\mu\varpi i\right)^2}}{1-\frac{\alpha^2}{\left(\left(m-\frac{1}{2}\right)\omega-\left(\mu-\frac{1}{2}\right)\varpi i\right)^2}},$$

$$F\alpha=\prod_{\mu=1}^{\infty}\left(1+\frac{\alpha^2}{\left(\mu-\frac{1}{2}\right)^2\varpi^2}\right)$$

$$\times\prod_{m=1}^{\infty}\prod_{\mu=1}^{\infty}\frac{1-\frac{\alpha^2}{\left(m\omega+\left(\mu-\frac{1}{2}\right)\varpi i\right)^2}}{1-\frac{\alpha^2}{\left(\left(m-\frac{1}{2}\right)\omega+\left(\mu-\frac{1}{2}\right)\varpi i\right)^2}}\,\frac{1-\frac{\alpha^2}{\left(m\omega-\left(\mu-\frac{1}{2}\right)\varpi i\right)^2}}{1-\frac{\alpha^2}{\left(\left(m-\frac{1}{2}\right)\omega-\left(\mu-\frac{1}{2}\right)\varpi i\right)^2}}.$$

Abel also writes these formulae in a real form.

The Eulerian products for $\sin y$ and $\cos y$ lead to

$$\prod_{\mu=1}^{\infty}\frac{1-\frac{z^2}{\mu^2\pi^2}}{1-\frac{y^2}{\left(\mu-\frac{1}{2}\right)^2\pi^2}}=\frac{\sin z}{z\cos y}\quad\text{and}\quad\prod_{\mu=1}^{\infty}\frac{1-\frac{z^2}{\left(\mu-\frac{1}{2}\right)^2\pi^2}}{1-\frac{y^2}{\left(\mu-\frac{1}{2}\right)^2\pi^2}}=\frac{\cos z}{\cos y}$$

and this permits to transform the double products of Abel's formulae in simple products:

$$\varphi\alpha=\frac{\varpi}{\pi}\frac{\sin\left(\alpha\frac{\pi i}{\varpi}\right)}{i}\prod_{m=1}^{\infty}\left(1-\frac{\alpha^2}{m^2\omega^2}\right)$$

$$\times \prod_{m=1}^{\infty} \frac{\sin(\alpha + m\omega)\frac{\pi i}{\varpi} \sin(\alpha - m\omega)\frac{\pi i}{\varpi} \cos^2\left(m - \frac{1}{2}\right)\omega\frac{\pi i}{\varpi}}{\cos\left(\alpha + \left(m - \frac{1}{2}\right)\omega\right)\frac{\pi i}{\varpi} \cos\left(\alpha - \left(m - \frac{1}{2}\right)\omega\right)\frac{\pi i}{\varpi} \sin^2 m\omega\frac{\pi i}{\varpi}}$$
$$\times \frac{\left(m\omega\frac{\pi i}{\varpi}\right)^2}{(\alpha + m\omega)(\alpha - m\omega)\frac{\pi^2 i^2}{\varpi^2}}$$
$$= \frac{\varpi}{\pi} \frac{\sin\frac{\alpha}{\varpi}\pi i}{i} \prod_{m=1}^{\infty} \frac{1 - \frac{\sin^2 \alpha\frac{\pi}{\varpi} i}{\sin^2 m\omega\frac{\pi}{\varpi} i}}{1 - \frac{\sin^2 \frac{\pi}{\varpi} i}{\cos^2\left(m - \frac{1}{2}\right)\omega\frac{\pi}{\varpi} i}}$$
$$= \frac{1}{2}\frac{\varpi}{\pi}\left(h^{\frac{\alpha}{\varpi}\pi} - h^{-\frac{\alpha}{\varpi}\pi}\right) \prod_{m=1}^{\infty} \frac{1 - \left(\frac{h^{\frac{\alpha}{\varpi}\pi} - h^{-\frac{\alpha}{\varpi}\pi}}{h^{m\frac{\omega}{\varpi}\pi} - h^{-m\frac{\omega}{\varpi}\pi}}\right)^2}{1 + \left(\frac{h^{\frac{\alpha}{\varpi}\pi} - h^{-\frac{\alpha}{\varpi}\pi}}{h^{\left(m-\frac{1}{2}\right)\frac{\omega}{\varpi}\pi} - h^{-\left(m-\frac{1}{2}\right)\frac{\omega}{\varpi}\pi}}\right)}$$
$$= \frac{\omega}{\pi} \sin\frac{\alpha\pi}{\omega} \prod_{m=1}^{\infty} \frac{1 + \frac{4\sin^2\frac{\alpha\pi}{\omega}}{\left(h^{\frac{m\varpi\pi}{\omega}} - h^{-\frac{m\varpi\pi}{\omega}}\right)^2}}{1 - \frac{4\sin^2\frac{\alpha\pi}{\omega}}{\left(h^{\frac{(2m-1)\varpi\pi}{2\omega}} - h^{-\frac{(2m-1)\varpi\pi}{2\omega}}\right)^2}} \tag{85}$$

where $h = 2.712818\ldots$ is the basis of natural logarithms. In the same way, he obtains

$$F\alpha = \prod_{m=1}^{\infty} \frac{1 + \frac{4\sin^2\frac{\alpha\pi}{\omega}}{\left(h^{\frac{(2m+1)\varpi\pi}{\omega}} - h^{-\frac{(2m+1)\varpi\pi}{\omega}}\right)^2}}{1 - \frac{4\sin^2\frac{\alpha\pi}{\omega}}{\left(h^{\frac{(2m+1)\varpi\pi}{\omega}} - h^{-\frac{(2m+1)\varpi\pi}{\omega}}\right)^2}},$$
$$f\alpha = \cos\frac{\alpha\pi}{\omega} \prod_{m=1}^{\infty} \frac{1 - \frac{4\sin^2\frac{\alpha\pi}{\omega}}{\left(h^{\frac{m\varpi\pi}{\omega}} + h^{-\frac{m\varpi\pi}{\omega}}\right)^2}}{1 - \frac{4\sin^2\frac{\alpha\pi}{\omega}}{\left(h^{\frac{(2m-1)\varpi\pi}{2\omega}} + h^{-\frac{(2m-1)\varpi\pi}{2\omega}}\right)^2}}.$$

These expansions were known to Gauss and they were independently discovered by Jacobi, who used a passage to the limit in the formulae of transformation for the elliptic functions.

The expansion of $\frac{1}{\text{ch} y}$ in simple fractions gives

$$\sum_{\mu=1}^{\infty} (-1)^{\mu} \frac{(2\mu + 1)\varpi}{\left(\alpha \pm \left(m + \frac{1}{2}\right)\omega\right)^2 + \left(\mu + \frac{1}{2}\right)^2 \varpi^2}$$
$$= \frac{2\pi}{\varpi} \frac{1}{h^{\left(\alpha \pm \left(m + \frac{1}{2}\right)\omega\right)\frac{\pi}{\varpi}} + h^{-\left(\alpha \pm \left(m + \frac{1}{2}\right)\omega\right)\frac{\pi}{\varpi}}}$$

which permits to transform the formulae (84) in simple series. Thus

$$\varphi\alpha = \frac{2}{ec}\frac{\pi}{\varpi}\sum_{m=0}^{\infty}(-1)^m\left(\frac{1}{h^{\left(\alpha-\left(m+\frac{1}{2}\right)\omega\right)\frac{\pi}{\varpi}}+h^{-\left(\alpha-\left(m+\frac{1}{2}\right)\omega\right)\frac{\pi}{\varpi}}}\right.$$
$$\left.-\frac{1}{h^{\left(\alpha+\left(m+\frac{1}{2}\right)\omega\right)\frac{\pi}{\varpi}}+h^{-\left(\alpha+\left(m+\frac{1}{2}\right)\omega\right)\frac{\pi}{\varpi}}}\right)$$
$$= \frac{2}{ec}\frac{\pi}{\varpi}\sum_{m=0}^{\infty}(-1)^m\frac{\left(h^{\frac{\alpha\pi}{\varpi}}-h^{-\frac{\alpha\pi}{\varpi}}\right)\left(h^{\left(m+\frac{1}{2}\right)\frac{\omega\pi}{\varpi}}-h^{-\left(m+\frac{1}{2}\right)\frac{\omega\pi}{\varpi}}\right)}{h^{\frac{2\alpha\pi}{\varpi}}+h^{-\frac{2\alpha\pi}{\varpi}}+h^{(2m+1)\frac{\omega\pi}{\varpi}}+h^{-(2m+1)\frac{\omega\pi}{\varpi}}}$$
$$= \frac{4}{ec}\frac{\pi}{\omega}\sum_{m=0}^{\infty}(-1)^m\frac{\sin\frac{\alpha\pi}{\varpi}\cdot\left(h^{\left(m+\frac{1}{2}\right)\frac{\varpi\pi}{\omega}}-h^{-\left(m+\frac{1}{2}\right)\frac{\varpi\pi}{\omega}}\right)}{h^{(2m+1)\frac{\varpi\pi}{\omega}}+2\cos 2\alpha\frac{\pi}{\omega}+h^{-(2m+1)\frac{\varpi\pi}{\omega}}} \qquad (86)$$

and

$$F\alpha = \frac{2}{c}\frac{\pi}{\varpi}\sum_{m=0}^{\infty}\frac{\left(h^{\frac{\alpha\pi}{\varpi}}+h^{-\frac{\alpha\pi}{\varpi}}\right)\left(h^{\left(m+\frac{1}{2}\right)\frac{\omega\pi}{\varpi}}+h^{-\left(m+\frac{1}{2}\right)\frac{\omega\pi}{\varpi}}\right)}{h^{\frac{2\alpha\pi}{\varpi}}+h^{-\frac{2\alpha\pi}{\varpi}}+h^{(2m+1)\frac{\omega\pi}{\varpi}}+h^{-(2m+1)\frac{\omega\pi}{\varpi}}},$$
$$f\alpha = \frac{4}{e}\frac{\pi}{\omega}\sum_{m=0}^{\infty}\frac{\cos\frac{\alpha\pi}{\omega}\cdot\left(h^{\left(m+\frac{1}{2}\right)\frac{\varpi\pi}{\omega}}+h^{-\left(m+\frac{1}{2}\right)\frac{\varpi\pi}{\omega}}\right)}{h^{(2m+1)\frac{\varpi\pi}{\omega}}+2\cos 2\alpha\frac{\pi}{\omega}+h^{-(2m+1)\frac{\varpi\pi}{\omega}}}.$$

In the lemniscatic case, where $e = c = 1$, one has $\omega = \varpi$ and these expansions take a simpler form

$$\varphi\left(\alpha\frac{\omega}{2}\right) = 2\frac{\pi}{\omega}\left(\frac{h^{\frac{\alpha\pi}{2}}-h^{-\frac{\alpha\pi}{2}}}{h^{\frac{\pi}{2}}+h^{-\frac{\pi}{2}}}-\frac{h^{\frac{3\alpha\pi}{2}}-h^{-\frac{3\alpha\pi}{2}}}{h^{\frac{3\pi}{2}}+h^{-\frac{3\pi}{2}}}+\frac{h^{\frac{5\alpha\pi}{2}}-h^{-\frac{5\alpha\pi}{2}}}{h^{\frac{5\pi}{2}}+h^{-\frac{5\pi}{2}}}-\dots\right)$$
$$= \frac{4\pi}{\omega}\left(\sin\left(\alpha\frac{\pi}{2}\right)\frac{h^{\frac{\pi}{2}}}{1+h^{\pi}}-\sin\left(3\alpha\frac{\pi}{2}\right)\frac{h^{\frac{3\pi}{2}}}{1+h^{3\pi}}+\sin\left(5\alpha\frac{\pi}{2}\right)\frac{h^{\frac{5\pi}{2}}}{1+h^{5\pi}}-\dots\right),$$
$$F\left(\alpha\frac{\omega}{2}\right) = 2\frac{\pi}{\omega}\left(\frac{h^{\frac{\alpha\pi}{2}}+h^{-\frac{\alpha\pi}{2}}}{h^{\frac{\pi}{2}}-h^{-\frac{\pi}{2}}}-\frac{h^{\frac{3\alpha\pi}{2}}+h^{-\frac{3\alpha\pi}{2}}}{h^{\frac{3\pi}{2}}-h^{-\frac{3\pi}{2}}}+\frac{h^{\frac{5\alpha\pi}{2}}+h^{-\frac{5\alpha\pi}{2}}}{h^{\frac{5\pi}{2}}-h^{-\frac{5\pi}{2}}}-\dots\right),$$
$$f\left(\alpha\frac{\omega}{2}\right) = \frac{4\pi}{\omega}\left(\cos\left(\alpha\frac{\pi}{2}\right)\frac{h^{\frac{\pi}{2}}}{h^{\pi}-1}-\cos\left(3\alpha\frac{\pi}{2}\right)\frac{h^{\frac{3\pi}{2}}}{h^{3\pi}-1}+\cos\left(5\alpha\frac{\pi}{2}\right)\frac{h^{\frac{5\pi}{2}}}{h^{5\pi}-1}-\dots\right)$$

and, taking $\alpha = 0$, $\frac{\omega}{2} = 2\pi\left(\frac{h^{\frac{\pi}{2}}}{h^{\pi}-1}-\frac{h^{\frac{3\pi}{2}}}{h^{3\pi}-1}+\frac{h^{\frac{5\pi}{2}}}{h^{5\pi}-1}-\dots\right) = \int_0^1\frac{dx}{\sqrt{1-x^4}}$,

$$\frac{\omega^2}{4} = \pi^2\left(\frac{h^{\frac{\pi}{2}}}{h^{\pi}+1}-3\frac{h^{\frac{3\pi}{2}}}{h^{3\pi}+1}+5\frac{h^{\frac{5\pi}{2}}}{h^{5\pi}+1}-\dots\right).$$

The second part of Abel's memoir, beginning with paragraph VIII (p. 352–362), was published in 1828. This paragraph is devoted to the algebraic solution of the equation $P_n = 0$ which gives $\varphi\left(\frac{\omega}{n}\right)$ in the lemniscatic case and for n a prime number of the form $4\nu + 1$. Abel announces that there is an infinity of other cases where the equation $P_n = 0$ is solvable by radicals.

Here, by the addition theorem (73), $\varphi(m + \mu i)\delta = \frac{\varphi(m\delta) f(\mu\delta) F(\mu\delta) + i\varphi(\mu\delta) f(m\delta) F(m\delta)}{1-\varphi^2(m\delta)\varphi^2(\mu\delta)}$ $= \varphi\delta T$ where T is a rational function of $(\varphi\delta)^2$ because of the formulae of multiplication. One says that there exists a *complex multiplication*. Putting $\varphi\delta = x$, we have $\varphi(m + \mu i)\delta = x\psi(x^2)$. Now $\varphi(\delta i) = i\varphi\delta = ix$ and $\varphi(m + \mu i)i\delta = i\varphi(m + \mu i)\delta = ix\psi(-x^2)$ and this shows that $\psi(-x^2) = \psi(x^2)$. In other words, ψ is an even function and T is a rational function of x^4. For instance

$$\varphi(2 + i)\delta = \frac{\varphi(2\delta) f\delta \cdot F\delta + i\varphi\delta \cdot f(2\delta) F(2\delta)}{1 - (\varphi 2\delta)^2 \varphi^2\delta},$$

where $\varphi(2\delta) = \frac{2x\sqrt{1-x^4}}{1+x^4}$, $f\delta = \sqrt{1 - x^2}$, $F\delta = \sqrt{1 + x^2}$, $f(2\delta) = \frac{1-2x^2-x^4}{1+x^4}$ and $F(2\delta) = \frac{1+2x^2-x^4}{1+x^4}$. Thus $\varphi(2+i)\delta = xi\frac{1-2i-x^4}{1-(1-2i)x^4}$. Gauss had already discovered the complex multiplication of lemniscatic functions and the fact that it made possible the algebraic solution of the division of the periods. He made an allusion to this fact in the introduction to the seventh section of his *Disquisitiones arithmeticae*, but never publish anything on the subject. We have explained this algebraic solution in our §3.

The ninth paragraph of Abel's memoir (p. 363–377) deals with the transformation of elliptic functions. The transformation of order 2 was known since Landen (1775) and Lagrange (1784) and Legendre made an extensive suty of it in his *Exercices de calcul integral*. Later, in 1824, Legendre discovered another transformation, of order 3, which Jacobi rediscovered in 1827 together with a new transformation, of order 5. Then Jacobi announced the existence of transformations of any orders, but he was able to prove this existence only in 1828, using the idea of inversion of the elliptic integrals which came from Abel. Independently from Jacobi, Abel built the theory of transformations. Here is his statement:

"If one designates by α the quantity $\frac{(m+\mu)\omega+(m-\mu)\varpi i}{2n+1}$, where at least one of the two integers m and μ is relatively prime with $2n + 1$, one has

$$\int \frac{dy}{\sqrt{(1 - c_1^2 y^2)(1 + e_1^2 y^2)}} = \pm a \int \frac{dx}{\sqrt{(1 - c^2 x^2)(1 + e^2 x^2)}} \tag{87}$$

where $y = f \cdot x \frac{(\varphi^2\alpha - x^2)(\varphi^2 2\alpha - x^2)\cdots(\varphi^2 n\alpha - x^2)}{(1+e^2c^2\varphi^2\alpha \cdot x^2)(1+e^2c^2\varphi^2 2\alpha \cdot x^2)\cdots(1+e^2c^2\varphi^2 n\alpha \cdot x^2)}$,

$$\frac{1}{c_1} = \frac{f}{c}\left(\varphi\left(\frac{\omega}{2} + \alpha\right)\varphi\left(\frac{\omega}{2} + 2\alpha\right)\cdots\varphi\left(\frac{\omega}{2} + n\alpha\right)\right)^2,$$

$$\frac{1}{e_1} = \frac{f}{e}\left(\varphi\left(\frac{\varpi i}{2} + \alpha\right)\varphi\left(\frac{\varpi i}{2} + 2\alpha\right)\cdots\varphi\left(\frac{\varpi i}{2} + n\alpha\right)\right)^2,$$

$$a = f(\varphi\alpha \cdot \varphi 2\alpha \cdot \varphi 3\alpha \cdots \varphi n\alpha)^2.\text{"} \tag{88}$$

Here f is indeterminate and e^2, c^2 may be positive or negative. By (80) (periodicity), we have $\varphi(\theta+(2n+1)\alpha)=\varphi\theta$ or $\varphi(\theta+(n+1)\alpha)=\varphi(\theta-n\alpha)$. Now if

$$\varphi_1\theta=\varphi\theta+\varphi(\theta+\alpha)+\ldots+\varphi(\theta+2n\alpha),$$

we have $\varphi_1(\theta+\alpha)=\varphi_1\theta$ and $\varphi_1\theta$ admits the period α. This function may be written

$$\begin{aligned}\varphi_1\theta=&\varphi\theta+\varphi(\theta+\alpha)+\varphi(\theta-\alpha)+\varphi(\theta+2\alpha)+\varphi(\theta-2\alpha)\\&+\ldots+\varphi(\theta+n\alpha)+\varphi(\theta-n\alpha)\\=&\varphi\theta+\frac{2\varphi\theta\cdot f\alpha\cdot F\alpha}{1+e^2c^2\varphi^2\alpha\cdot\varphi^2\theta}+\frac{2\varphi\theta\cdot f2\alpha\cdot F2\alpha}{1+e^2c^2\varphi^2 2\alpha\cdot\varphi^2\theta}\\&+\ldots+\frac{2\varphi\theta\cdot fn\alpha\cdot Fn\alpha}{1+e^2c^2\varphi^2 n\alpha\cdot\varphi^2\theta},\end{aligned}\tag{89}$$

a rational function ψx of $x=\varphi\theta$. Note that the auxiliary function $\varphi_1\theta$ used to solve the equation of division in the first part was precisely of this type (see §3).

For any ε, $R=\left(1-\frac{\psi x}{\varphi_1\varepsilon}\right)(1+e^2c^2\varphi^2\alpha x^2)\ldots(1+e^2c^2\varphi^2 n\alpha x^2)$ is a polynomial of degree $2n+1$ in x. It is annihilated by $x=\varphi\varepsilon$ and so by $x=\varphi(\varepsilon+m\alpha)$, m any integer. Since $\varphi\varepsilon$, $\varphi(\varepsilon+\alpha)$, $\varphi(\varepsilon+2\alpha)$, … , $\varphi(\varepsilon+2n\alpha)$ are all different, they are the roots of R and

$$R=A\left(1-\frac{x}{\varphi\varepsilon}\right)\left(1-\frac{x}{\varphi(\varepsilon+\alpha)}\right)\cdots\left(1-\frac{x}{\varphi(\varepsilon+2n\alpha)}\right)\tag{90}$$

where A is found to be 1 by making $x=0$. Multiplying by $\varphi\varepsilon$ and then making $\varepsilon=0$, we obtain

$$\begin{aligned}\psi x=&gx\frac{\left(1-\frac{x}{\varphi\alpha}\right)\left(1-\frac{x}{\varphi2\alpha}\right)\cdots\left(1-\frac{x}{\varphi2n\alpha}\right)}{(1+e^2c^2\varphi^2\alpha\cdot x^2)\cdots(1+e^2c^2\varphi^2 n\alpha\cdot x^2)}\\=&gx\frac{\left(1-\frac{x^2}{\varphi^2\alpha}\right)\left(1-\frac{x^2}{\varphi^2 2\alpha}\right)\cdots\left(1-\frac{x^2}{\varphi^2 2n\alpha}\right)}{(1+e^2c^2\varphi^2\alpha\cdot x^2)\cdots(1+e^2c^2\varphi^2 n\alpha\cdot x^2)}\end{aligned}\tag{91}$$

where $g=1+2f\alpha\cdot F\alpha+2f2\alpha\cdot F2\alpha+\ldots 2fn\alpha\cdot Fn\alpha$ is the value of $\frac{\varphi_1\varepsilon}{\varphi\varepsilon}$ for $\varepsilon=0$.

Doing $\varepsilon=\frac{\omega}{2}$ in R , we have

$$\begin{aligned}&1-\frac{\psi x}{\varphi_1\frac{\omega}{2}}\\&=\left(1-\frac{x}{\varphi\frac{\omega}{2}}\right)\left(1-\frac{x}{\varphi\left(\frac{\omega}{2}+\alpha\right)}\right)\left(1-\frac{x}{\varphi\left(\frac{\omega}{2}+2\alpha\right)}\right)\cdots\left(1-\frac{x}{\varphi\left(\frac{\omega}{2}+2n\alpha\right)}\right)\frac{1}{\rho}\\&=(1-cx)\left(1-\frac{x}{\varphi\left(\frac{\omega}{2}+\alpha\right)}\right)^2\left(1-\frac{x}{\varphi\left(\frac{\omega}{2}+2\alpha\right)}\right)^2\cdots\left(1-\frac{x}{\varphi\left(\frac{\omega}{2}+n\alpha\right)}\right)^2\frac{1}{\rho}\end{aligned}$$

where $\rho=(1+e^2c^2\varphi^2\alpha\cdot x^2)(1+e^2c^2\varphi^2 2\alpha\cdot x^2)\cdots(1+e^2c^2\varphi^2 n\alpha\cdot x^2)$. Changing x in $-x$, we have

$$1+\frac{\psi x}{\varphi_1\frac{\omega}{2}}=(1+cx)\left(1+\frac{x}{\varphi\left(\frac{\omega}{2}+\alpha\right)}\right)^2\left(1+\frac{x}{\varphi\left(\frac{\omega}{2}+2\alpha\right)}\right)^2$$
$$\times\cdots\left(1+\frac{x}{\varphi\left(\frac{\omega}{2}+n\alpha\right)}\right)^2\frac{1}{\rho}.$$

Abel puts $y=k\psi x$, $c_1=\frac{1}{k\varphi_1\frac{\omega}{2}}$ (k a constant),

$$t=\left(1-\frac{x}{\varphi\left(\frac{\omega}{2}+\alpha\right)}\right)\left(1-\frac{x}{\varphi\left(\frac{\omega}{2}+2\alpha\right)}\right)\cdots\left(1-\frac{x}{\varphi\left(\frac{\omega}{2}+n\alpha\right)}\right),$$
$$t_1=\left(1+\frac{x}{\varphi\left(\frac{\omega}{2}+\alpha\right)}\right)\left(1+\frac{x}{\varphi\left(\frac{\omega}{2}+2\alpha\right)}\right)\cdots\left(1+\frac{x}{\varphi\left(\frac{\omega}{2}+n\alpha\right)}\right)$$

in order to have $1-c_1y=(1-cx)\frac{t^2}{\rho}$ and $1+c_1y=(1+cx)\frac{t_1^2}{\rho}$.

In the same way, $1\mp e_1iy=(1-eix)\frac{s_1^2}{\rho}$, $1\pm e_1iy=(1+eix)\frac{s^2}{\rho}$ where $e_1=\pm\frac{i}{k\varphi_1\left(\frac{\varpi}{2}i\right)}$ and

$$s=\left(1-\frac{x}{\varphi\left(\frac{\varpi}{2}i+\alpha\right)}\right)\left(1-\frac{x}{\varphi\left(\frac{\varpi}{2}i+2\alpha\right)}\right)\cdots\left(1-\frac{x}{\varphi\left(\frac{\varpi}{2}i+n\alpha\right)}\right),$$
$$s_1=\left(1+\frac{x}{\varphi\left(\frac{\varpi}{2}i+\alpha\right)}\right)\left(1+\frac{x}{\varphi\left(\frac{\varpi}{2}i+2\alpha\right)}\right)\cdots\left(1+\frac{x}{\varphi\left(\frac{\varpi}{2}i+n\alpha\right)}\right).$$

Thus $\sqrt{(1-c_1^2y^2)(1+e_1^2y^2)}=\pm\frac{tt_1ss_1}{\rho^2}\sqrt{(1-c^2x^2)(1+e^2x^2)}$. Now $dy=\frac{P}{\rho^2}dx$ where P is a polynomial of degree $4n$. Differentiating $1-c_1y=(1-cx)\frac{t^2}{\rho}$, we see that
$$P=\frac{t}{c_1}\left(ct\rho-(1-cx)\left(2\rho\frac{dt}{dx}-t\frac{d\rho}{dx}\right)\right)$$
is divisible by t and, in the same manner, it is divisible by t_1, s and s_1. Since these four polynomials of degree n cannot have any common factor, it results that $\frac{P}{tt_1ss_1}$ is a constant a and that $\frac{dy}{\sqrt{(1-c_1^2y^2)(1+e_1^2y^2)}}=\pm a\frac{dx}{\sqrt{(1-c^2x^2)(1+e^2x^2)}}$. When $x=0$, $t=t_1=s=s_1=1=\rho$ and $P=\frac{dy}{dx}=k\psi'(0)=kg$.

According to (90), the coefficient of x^{2n+1} in R is $-\frac{1}{\varphi\varepsilon\cdot\varphi(\varepsilon+\alpha)\cdots\varphi(\varepsilon+n\alpha)}$ and, comparing with (91) it is equal to $-\frac{(-1)^n}{\varphi_1\varepsilon}\frac{g}{(\varphi\alpha\cdot\varphi2\alpha\cdots\varphi n\alpha)^2}$. Thus

$$\varphi_1\varepsilon=\frac{(-1)^ng}{(\varphi\alpha\cdot\varphi2\alpha\cdots\varphi n\alpha)^2}\varphi\varepsilon\cdot\varphi(\varepsilon+\alpha)\cdots\varphi(\varepsilon+n\alpha).$$

Now (89) and (91) give two expressions for the limit of $\frac{\psi x}{x}$ for x infinite:

$$1 \text{ and } \frac{g(-1)^n}{(ce)^{2n}(\varphi\alpha\cdot\varphi2\alpha\cdots\varphi n\alpha)^4}.$$

Thus, by comparison, $g=(-1)^n(ec)^{2n}(\varphi\alpha\varphi2\alpha\cdots\varphi n\alpha)^4$ and

$$\varphi_1\varepsilon=(ec)^{2n}(\varphi\alpha\varphi2\alpha\cdots\varphi n\alpha)^2\varphi\varepsilon\varphi(\varepsilon+\alpha)\cdots\varphi(\varepsilon+2n\alpha).$$

In particular

$$\varphi_1\left(\frac{\omega}{2}\right)=\frac{1}{kc_1}=(ec)^{2n}\delta^2\varphi\left(\frac{\omega}{2}\right)\varphi\left(\frac{\omega}{2}+\alpha\right)\cdots\varphi\left(\frac{\omega}{2}+2n\alpha\right),$$
$$\varphi_1\left(\frac{\varpi}{2}i\right)=\frac{\pm i}{ke_1}=(ec)^{2n}\delta^2\varphi\left(\frac{\varpi}{2}i\right)\varphi\left(\frac{\varpi}{2}i+\alpha\right)\cdots\varphi\left(\frac{\varpi}{2}i+2n\alpha\right)$$

where $\delta=\varphi\alpha\cdot\varphi2\alpha\cdots\varphi n\alpha$g The values (88) of the statement for $\frac{1}{c_1}$ and $\frac{1}{e_1}$ result if we put $f=k(e^2c^2)^n\delta^2$ and, as $\varphi\left(\frac{\omega}{2}+\alpha\right)\varphi\left(\frac{\varpi}{2}i+\alpha\right)=\frac{i}{ec}$ (cf. (76)), we obtain $c_1e_1=\pm\frac{(-1)^n(ec)^{2n+1}}{f^2}$. On the other hand

$$\pm\frac{e_1}{c_1}=(-1)^n\frac{e}{c}(ec)^{2n}\left(\varphi\left(\frac{\omega}{2}+\alpha\right)\varphi\left(\frac{\omega}{2}+2\alpha\right)\cdots\varphi\left(\frac{\omega}{2}+n\alpha\right)\right)^4,$$
$$\pm\frac{c_1}{e_1}=(-1)^n\frac{c}{e}(ec)^{2n}\left(\varphi\left(\frac{\varpi}{2}i+\alpha\right)\varphi\left(\frac{\varpi}{2}i+2\alpha\right)\cdots\varphi\left(\frac{\varpi}{2}i+n\alpha\right)\right)^4$$

and $a=kg=(-1)^nf\cdot\delta^2$.

Using $\left(\varphi\left(\frac{\omega}{2}+\alpha\right)\right)^2=\frac{1}{c^2}\frac{1-c^2\varphi^2\alpha}{1+c^2\varphi^2\alpha}$ and $\left(\varphi\left(\frac{\varpi}{2}i+\alpha\right)\right)^2=-\frac{1}{e^2}\frac{1+e^2\varphi^2\alpha}{1-e^2\varphi^2\alpha}$ (cf. (76)), one transforms the expressions of $\frac{1}{c_1}$ and $\frac{1}{e_1}$ in rational symmetric functions of $\varphi\alpha$, $\varphi2\alpha,\ldots,\varphi n\alpha$. Reasoning as in his §V for the equation $P_{2n+1}=0$, Abel deduces that, when $2n+1$ is a prime number, c_1 and e_1 are determined by an equation of degree $2n+2$ (the 'modular equation' as it was called later). Now such an equation has roots not necessarily real and Abel says that the theory must be extended to the case of moduli c, e complex numbers.

When c and e are real, the only values of α giving c_1 and e_1 real are $\frac{2m\omega}{2n+1}$ and $\frac{2\mu\varpi i}{2n+1}$. The first value gives

$$\frac{1}{c_1}=\frac{f}{c}\left(\varphi\left(\frac{1}{2n+1}\frac{\omega}{2}\right)\varphi\left(\frac{3}{2n+1}\frac{\omega}{2}\right)\cdots\varphi\left(\frac{2n-1}{2n+1}\frac{\omega}{2}\right)\right)^2,$$

$$\frac{e_1}{c_1}=\pm(-1)^n\frac{e}{c}(ec)^{2n}\left(\varphi\left(\frac{1}{2n+1}\frac{\omega}{2}\right)\varphi\left(\frac{3}{2n+1}\frac{\omega}{2}\right)\cdots\varphi\left(\frac{2n-1}{2n+1}\frac{\omega}{2}\right)\right)^4.$$

Abel explains in particular the case in which $c=c_1=1,\pm(-1)^n=1$ and $0<e<1$. Then e_1 is very small when $2n+1$ is large. Abel carefully studies the sign in (87). Since ρ^2 is positive for x real, this sign is that of $tt_1ss_1\sqrt{\frac{1-x^2}{1-y^2}}$.

Now ss_1 is easily seen to be positive and the sign we are looking for is that of $tt_1 = \left(1 - \frac{x^2}{\varphi^2\left(\frac{1}{2n+1}\frac{\omega}{2}\right)}\right)\left(1 - \frac{x^2}{\varphi^2\left(\frac{3}{2n+1}\frac{\omega}{2}\right)}\right)\cdots\left(1 - \frac{x^2}{\varphi^2\left(\frac{2n-1}{2n+1}\frac{\omega}{2}\right)}\right)$ for the radical is positive. For instance, when $-\varphi\left(\frac{1}{2n+1}\frac{\omega}{2}\right) \leq x \leq \varphi\left(\frac{1}{2n+1}\frac{\omega}{2}\right)$ the sign is + and we get $(-1)^n a = \frac{4n+2}{\omega}\int_0^1 \frac{dy}{\sqrt{(1-y^2)(1+e_1^2y^2)}}$ by doing $x = \varphi\left(\frac{1}{2n+1}\frac{\omega}{2}\right)$ to which corresponds $y = (-1)^n$. If we neglect e_1^2, this gives approximately $(-1)^n a = (2n+1)\frac{\pi}{\omega}$ and

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1+e^2x^2)}} = \frac{(-1)^n\omega}{(2n+1)\pi}\arcsin y$$

for $y = (-1)^n(2n+1)\frac{\pi}{\omega}x\frac{\left(1-\frac{x^2}{\varphi^2\left(\frac{\omega}{2n+1}\right)}\right)\cdots\left(1-\frac{x^2}{\varphi^2\left(\frac{n\omega}{2n+1}\right)}\right)}{\left(1+e^2\varphi^2\left(\frac{\omega}{2n+1}\right)x^2\right)\cdots\left(1+e^2\varphi^2\left(\frac{n\omega}{2n+1}\right)x^2\right)}$. Abel also explains the effect of the other real transformation $\alpha = \frac{2\mu\varpi i}{2n+1}$. He states that every possible transformation is obtained by combining the transformations of order 2^k studied by Legendre with his new transformations. He will publish a proof of this statement in his *Précis d'une théorie des fonctions elliptiques* (1829).

The last paragraph of the *Recherches* (p. 377–388) is devoted to the study of the differential equation $\frac{dy}{\sqrt{(1-y^2)(1+\mu y^2)}} = a\frac{dx}{\sqrt{(1-x^2)(1+\mu x^2)}}$ and, in particular to the cases in which there is complex multiplication. Abel states two theorems: "I. Supposing a real and the equation algebraically integrable, it is necessary that a be a rational number."

"II. Supposing *a imaginary* and the equation *algebraically* integrable, it is necessary that a be of the form $m \pm \sqrt{-1}\cdot\sqrt{n}$ where m and n are rational numbers. In this case, the quantity μ is not arbitrary; it must satisfy an equation which has an infinity of roots, real and imaginary. Each value of μ satisfy to the question."

Here Abel only considers a particular case, that in which $e_1 = \frac{1}{e}$ for the first real transformation. Thus we have

$$\frac{dy}{\sqrt{(1-y^2)(1+e^2y^2)}} = a\sqrt{-1}\frac{dx}{\sqrt{(1-x^2)(1+e^2x^2)}} \quad (\text{changing } y \text{ in } \frac{ey}{i}), \qquad (92)$$

where $y = \pm\sqrt{-1}e^n x\frac{\left(\varphi^2\left(\frac{\omega}{2n+1}\right)-x^2\right)\cdots\left(\varphi^2\left(\frac{n\omega}{2n+1}\right)-x^2\right)}{\left(1+e^2\varphi^2\left(\frac{\omega}{2n+1}\right)x^2\right)\cdots\left(1+e^2\varphi^2\left(\frac{n\omega}{2n+1}\right)x^2\right)}$, e being determined by $1 = e^{n+1}\left(\varphi\left(\frac{1}{2n+1}\frac{\omega}{2}\right)\cdots\varphi\left(\frac{2n-1}{2n+1}\frac{\omega}{2}\right)\right)^2$ and a by $a = \pm\frac{1}{e}\left(\frac{\varphi\left(\frac{\omega}{2n+1}\right)\cdots\varphi\left(\frac{n\omega}{2n+1}\right)}{\varphi\left(\frac{1}{2n+1}\frac{\omega}{2}\right)\cdots\varphi\left(\frac{2n-1}{2n+1}\frac{\omega}{2}\right)}\right)^2$.

From (92), Abel deduces $\frac{\varpi}{2} = \int_0^{\frac{1}{e}} \frac{dz}{\sqrt{(1+z^2)(1-e^2z^2)}} = a\frac{\omega}{4n+2}$ (integration from $x = 0$ to $x = \varphi\left(\frac{\omega}{4n+2}\right)$, $y = z\sqrt{-1}$) and $\frac{\omega}{2} = a\frac{\varpi}{2}$. Thus $a = \sqrt{2n+1} = \frac{\omega}{\varpi}$. For instance, when $n = 1$, $\frac{dy}{\sqrt{(1-y^2)(1+e^2y^2)}} = \sqrt{-3}\frac{dx}{\sqrt{(1-x^2)(1+e^2x^2)}}$ where

$y = \sqrt{-1}ex\frac{\varphi^2\left(\frac{\omega}{3}\right)-x^2}{1+e^2\varphi^2\left(\frac{\omega}{3}\right)x^2}$,

$$1 = e^2\left(\varphi\left(\frac{1}{3}\frac{\omega}{2}\right)\right)^2 = \frac{e^2 - e^2\varphi^2\left(\frac{\omega}{3}\right)}{1+e^2\varphi^2\left(\frac{\omega}{3}\right)} \text{ and } a = \frac{\varphi^2\left(\frac{\omega}{3}\right)}{\varphi^2\left(\frac{\omega}{6}\right)}\frac{1}{e} = \sqrt{3}.$$

This gives $\varphi\left(\frac{\omega}{3}\right) = \frac{\sqrt{3}}{e}$ and $e = \sqrt{3}+2, \varphi\left(\frac{\omega}{3}\right) = 2\sqrt{3}-3$. Changing x in $x\sqrt{2-\sqrt{3}}$ and y in $y\sqrt{2-\sqrt{3}}\sqrt{-1}$, Abel obtains $\frac{dy}{\sqrt{1-2\sqrt{3}y^2-y^4}} = \sqrt{3}\frac{dx}{\sqrt{1+2\sqrt{3}x^2-x^4}}$ where $y = x\frac{\sqrt{3}-x^2}{1+\sqrt{3}x^2}$.

For $n=2$, $\frac{dy}{\sqrt{(1-y^2)(1+e^2y^2)}} = \sqrt{-5}\frac{dx}{\sqrt{(1-x^2)(1+e^2x^2)}}$ where $y = \sqrt{-1}e^2x\frac{\varphi^2\left(\frac{\omega}{5}\right)-x^2}{1+e^2\varphi^2\left(\frac{\omega}{5}\right)x^2} \times \frac{\varphi^2\left(\frac{2\omega}{5}\right)-x^2}{1+e^2\varphi^2\left(\frac{2\omega}{5}\right).x^2}$, $1 = e^2\varphi^2\left(\frac{\omega}{10}\right)\varphi^2\left(\frac{3\omega}{10}\right)$, $\sqrt{5} = e^2\varphi^2\left(\frac{\omega}{5}\right)\varphi^2\left(\frac{2\omega}{5}\right)$. Using $\varphi^2\left(\frac{\omega}{10}\right) = \varphi^2\left(\frac{\omega}{2}-\frac{2\omega}{5}\right) = \frac{f^2\left(\frac{2\omega}{5}\right)}{F^2\left(\frac{2\omega}{5}\right)}$ and $\varphi^2\left(\frac{3\omega}{10}\right) = \varphi^2\left(\frac{\omega}{2}-\frac{\omega}{5}\right) = \frac{f^2\left(\frac{\omega}{5}\right)}{F^2\left(\frac{\omega}{5}\right)}$, Abel finally gets $-\frac{1}{e\sqrt{e}} = \frac{1}{e^2}\frac{1-e\sqrt{5}}{e-\sqrt{5}}$, which gives a cubic equation for e:

$$e^3 - 1 - (5+2\sqrt{5})e(e-1) = 0.$$

This equation has only one solution larger than 1, as e must be, $e = \left(\frac{\sqrt{5}+1}{2}+\sqrt{\frac{\sqrt{5}+1}{2}}\right)^2$. It is then easy to compute $\alpha = \varphi\left(\frac{\omega}{5}\right)$ and $\beta = \varphi\left(\frac{2\omega}{5}\right)$ for $\alpha^2\beta^2 = \frac{\sqrt{5}}{e^2}$ and

$$e^3 - 1 - e(e-1)\sqrt{5} = e^2(e+1)(\alpha^2+\beta^2).$$

Changing x in $\frac{x}{\sqrt{e}}$ and y in $\frac{y\sqrt{-1}}{\sqrt{e}}$, Abel obtains the equation

$$\frac{dy}{\sqrt{1-4\sqrt{2+\sqrt{5}}y^2-y^4}} = \sqrt{5}\frac{dx}{\sqrt{1+4\sqrt{2+\sqrt{5}}x^2-x^4}}$$

where $y = x\frac{\sqrt{5}-\sqrt{10+10\sqrt{5}}x^2+x^4}{1+\sqrt{10+10\sqrt{5}}x^2+\sqrt{5}x^4}$.

For higher orders n Abel says that the equation giving the singular modulus e is not necessarily algebraically solvable and he proposes an expansion of e in infinite series. He starts from (86) with $\alpha = \frac{\omega}{2}$, $\varphi\left(\frac{\omega}{2}\right) = \frac{1}{c} = 1$ and gets $e\omega = 4\pi\left(\frac{\rho}{\rho^2+1}+\frac{\rho^3}{\rho^6+1}+\frac{\rho^5}{\rho^{10}+1}+\dots\right)$ where $\rho = h^{\frac{\varpi}{\omega}\frac{\pi}{2}}$. With $\alpha = \frac{\varpi}{2}i$, $\varphi\left(\frac{\varpi}{2}i\right) = \frac{i}{e}$, Abel gets $\varpi = 4\pi\left(\frac{r}{r^2+1}+\frac{r^3}{r^6+1}+\frac{r^5}{r^{10}+1}+\dots\right)$ where $r = h^{\frac{\omega}{\varpi}\frac{\pi}{2}}$ and, since $\frac{\omega}{\varpi} = \sqrt{2n+1}$, $\omega = 4\pi\sqrt{2n+1}\left(\frac{h^{\frac{\pi}{2}\sqrt{2n+1}}}{h^{\pi\sqrt{2n+1}}+1}+\frac{h^{\frac{3\pi}{2}\sqrt{2n+1}}}{h^{3\pi\sqrt{2n+1}}+1}+\dots\right)$ and $e = \frac{4\pi}{\omega}\left(\frac{h^{\frac{\pi}{2}\frac{1}{\sqrt{2n+1}}}}{h^{\frac{\pi}{\sqrt{2n+1}}}+1}+\frac{h^{\frac{3\pi}{2}\frac{1}{\sqrt{2n+1}}}}{h^{\frac{3\pi}{\sqrt{2n+1}}}+1}+\dots\right)$.

At the end of this memoir, Abel explains how his theory of transformation gives the formulae published by Jacobi in 1827. Jacobi uses Legendre's notations, with a modulus k between 0 and 1 and the elliptic integral of the first kind $F(k,\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, so that if $\alpha = F(k,\theta)$, $\varphi\alpha = \sin\theta$ where φ is Abel's elliptic function with $c = 1$ and $e^2 = -k^2$. Writing $c_1 = 1$, $e_1^2 = -\lambda^2$, $\mu = \frac{(-1)^n}{a}$, $x = (-1)^n \sin\theta$, $y = \sin\psi$ and $2n+1 = p$, Abel's formula for the first real transformation takes the form $\int \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \pm\mu \int \frac{d\psi}{\sqrt{1-\lambda^2\sin^2\psi}} + C$, where

$$\lambda = k^{2n+1}(\sin\theta' \cdot \sin\theta''' \ldots \sin\theta^{(2n-1)})^4, \quad \mu = \left(\frac{\sin\theta' \cdot \sin\theta''' \cdots \sin\theta^{(2n-1)}}{\sin\theta'' \cdot \sin\theta'''' \cdots \sin\theta^{(2n)}}\right)^2$$

and $\sin\psi = \frac{k^{n+\frac{1}{2}}}{\sqrt{\lambda}} \sin\theta \frac{(\sin^2\theta''-\sin^2\theta)(\sin^2\theta''''-\sin^2\theta)\ldots(\sin^2\theta^{(2n)}-\sin^2\theta)}{(1-k^2\sin^2\theta''\cdot\sin^2\theta)(1-k^2\sin^2\theta''''\cdot\sin^2\theta)\ldots(1-k^2\sin^2\theta^{(2n)}\cdot\sin^2\theta)}$, the angles $\theta', \theta'', \ldots, \theta^{(2n)}$ being defined by $\sin\theta^{(m)} = \varphi\left(\frac{m}{p}\frac{\omega}{2}\right)$ or $F(k,\theta^{(m)}) = \frac{m}{p}\frac{\omega}{2}$. Since $c = c_1 = 1$, $\sqrt{\frac{1-y}{1+y}} = \frac{t}{t_1}\sqrt{\frac{1-x}{1+x}}$ or $\sqrt{\frac{1-\sin\psi}{1+\sin\psi}} = \sqrt{\frac{1-(-1)^n\sin\theta}{1+(-1)^n\sin\theta}} \frac{\sin\theta'-\sin\theta}{\sin\theta'+\sin\theta} \frac{\sin\theta'''+\sin\theta}{\sin\theta'''-\sin\theta} \ldots \frac{\sin\theta^{(2n-1)}+(-1)^n\sin\theta}{\sin\theta^{(2n-1)}-(-1)^n\sin\theta}$, relation which may be transformed in

$$\tan\left(45^\circ - \frac{1}{2}\psi\right) = \frac{\tan\frac{1}{2}(\theta'-\theta)}{\tan\frac{1}{2}(\theta'+\theta)} . \frac{\tan\frac{1}{2}(\theta'''+\theta)}{\tan\frac{1}{2}(\theta'''-\theta)} \ldots$$
$$\times \frac{\tan\frac{1}{2}(\theta^{(2n-1)}+(-1)^n\theta)}{\tan\frac{1}{2}(\theta^{(2n-1)}-(-1)^n\theta)} \tan\left(45^\circ - (-1)^n\frac{1}{2}\theta\right).$$

In 1828, Abel had begun the redaction of a second memoir to continue the *Recherches sur les fonctions elliptiques* (*Œuvres*, t. II, p. 244–253). Putting $\alpha = \frac{(m+\mu)\omega+(m-\mu)\varpi i}{2n+1}$ where m, μ and n are integers such that $m+\mu$, $m-\mu$ and $2n+1$ have no common divisor, and

$$\begin{aligned}\varphi_1\theta &= \varphi\theta \cdot \varphi(\alpha+\theta)\varphi(\alpha-\theta)\varphi(2\alpha+\theta)\varphi(2\alpha-\theta)\cdots\varphi(n\alpha+\theta)\varphi(n\alpha-\theta)\\ &= \varphi\theta \frac{\varphi^2\alpha-\varphi^2\theta}{1+e^2c^2\varphi^2\alpha\cdot\varphi^2\theta} \frac{\varphi^2 2\alpha-\varphi^2\theta}{1+e^2c^2\varphi^2 2\alpha\cdot\varphi^2\theta} \cdots \frac{\varphi^2 n\alpha-\varphi^2\theta}{1+e^2c^2\varphi^2 n\alpha\cdot\varphi^2\theta},\end{aligned}$$

Abel remarks that this function is rational in $\varphi\theta$ and invariant by $\theta \mapsto \theta \pm \alpha$. It results that the roots of the equation

$$\begin{aligned}0 &= x(\varphi^2\alpha - x^2)(\varphi^2 2\alpha - x^2)\cdots(\varphi^2 n\alpha - x^2)\\ &\quad -\varphi_1\theta(1+e^2c^2\varphi^2\alpha x^2)(1+e^2c^2\varphi^2 2\alpha x^2)\cdots(1+e^2c^2\varphi^2 n\alpha x^2)\end{aligned}$$

are $\varphi\theta, \varphi(\theta+\alpha), \ldots, \varphi(\theta+2n\alpha)$. Now let $\psi\theta$ be a rational function of these roots and suppose that it is invariant by $\theta \to \theta+\alpha$. Using the addition theorem (73), one sees that

$$\psi\theta = \psi_1\theta + \psi_2\theta \cdot f\theta \cdot F\theta$$

where $\psi_1\theta$ and $\psi_2\theta$ are rational in $\varphi\theta$ and $\psi_1\theta = \frac{1}{2}(\psi\theta + {}'\psi\theta)$, $\psi_2\theta \cdot f\theta \cdot F\theta = \frac{1}{2}(\psi\theta - {}'\psi\theta)$ where ${}'\psi\theta$ is the function deduced from $\psi\theta$ by changing α into $-\alpha$. One has ${}'\psi\theta = \psi_1\theta - \psi_2\theta \cdot f\theta \cdot F\theta$ and ${}'\psi\theta$ is invariant by $\theta \mapsto \theta + \alpha$. Thus $\psi_1\theta$ is invariant by $\theta \mapsto \theta + \alpha$ and it is thus a rational symmetric function of $\varphi\theta$, $\varphi(\theta + \alpha), \dots, \varphi(\theta + 2n\alpha)$, that is a rational function of $\varphi_1\theta$. In the same way, one shows that the square of $\psi_2\theta f\theta F\theta$ is a rational function of $\varphi_1\theta$, so that $\psi\theta = p \pm \sqrt{q'}$ where p, q' are rational functions of $\varphi_1\theta$. Let

$$\chi\theta = (\varphi\theta)^2\varphi(\theta + \alpha) + (\varphi(\theta + \alpha))^2\varphi(\theta + 2\alpha) + \dots$$
$$+(\varphi(\theta + (2n - 1)\alpha))^2\varphi(\theta + 2n\alpha) + (\varphi(\theta + 2n\alpha))^2\varphi\theta$$

and let ${}'\chi\theta$ be the function deduced from $\chi\theta$ by changing α into $-\alpha$. One has

$$\chi\theta = \chi_1\theta + \chi_2\theta \cdot f\theta \cdot F\theta, \quad {}'\chi\theta = \chi_1\theta - \chi_2\theta \cdot f\theta \cdot F\theta$$

where $\chi_1\theta$ and $\chi_2\theta$ are rational functions of $\varphi\theta$, and $\frac{1}{2}(\chi\theta - {}'\chi\theta) = \chi_2\theta \cdot f\theta \cdot F\theta = \pm\sqrt{r}$ where r is a rational function of $\varphi_1\theta$. Now $\frac{\psi_2^\theta}{\chi_2^\theta} = \frac{\psi\theta - {}'\psi\theta}{\chi\theta - {}'\chi\theta}$ is a rational function of $\varphi\theta$ invariant by $\theta \to \theta + \alpha$, so a rational function q of $\varphi_1\theta$ and $\frac{1}{2}(\psi\theta - {}'\psi\theta) = \pm q\sqrt{r}$, $\psi\theta = p \pm q\sqrt{r}$ where r does not depend of the function $\psi\theta$.

Abel proves that r is a polynomial in $\varphi_1\theta$, for if it had a pole $\varphi_1\delta$, we should have $\chi\delta - {}'\chi\delta = \frac{1}{0}$ which means that some $\varphi(\delta \pm \nu\alpha)$ would be infinite, but then $\varphi_1\delta$ would also be infinite, which is absurd. The expansions of our functions in decreasing powers of $x = \varphi\theta$ are $\varphi_1\theta = ax + \varepsilon$, $\chi\theta - {}'\chi\theta = Ax^2 + \varepsilon'$ where a, A are constant and ε and ε' contain powers of x respectively less than 1 and 2. If ν is the degree of r, the equation $r = \frac{1}{4}(\chi\theta - {}'\chi\theta)^2$ is rewritten $a'x^\nu + \dots = \frac{1}{4}A^2x^4 + \dots$ and it shows that $\nu = 4$. Since r must be annihilated by $\theta = \pm\frac{\omega}{2}, \pm\frac{\varpi}{2}i$, one has $r = C\left(1 - \left(\frac{\varphi_1\theta}{\varphi_1\frac{\omega}{2}}\right)^2\right)\left(1 - \left(\frac{\varphi_1\theta}{\varphi_1\frac{\varpi}{2}i}\right)^2\right)$ where C is a constant.

When $\psi\theta$ is a polynomial in $\varphi\theta, \varphi(\theta + \alpha), \dots, \varphi(\theta + 2n\alpha)$, the same reasoning shows that p and q are polynomials in $\varphi_1\theta$ of respective degrees ν and $\nu - 2$ where ν is the degree of $\psi\theta$ with respect to any one of the quantities $\varphi\theta$, $\varphi(\theta + \alpha), \dots, \varphi(\theta + 2n\alpha)$. If $\nu = 1$, one has $\psi\theta = A + B\varphi_1\theta$ where A and B are constants respectively determined by making $\theta = 0$ and $\theta = \frac{1}{0}$. For instance, let us put $\pi\theta = \varphi\theta \cdot \varphi(\theta + \nu_1\alpha)\varphi(\theta + \nu_2\alpha)\dots\varphi(\theta + \nu_\omega\alpha)$ and

$$P = \pi(\theta) + \pi(\theta + \alpha) + \pi(\theta + 2\alpha) + \dots + \pi(\theta + 2n\alpha)$$

where $\nu_1, \nu_2, \dots, \nu_\omega$ are distinct integers less than $2n + 1$. One has $A = \pi(\alpha) + \pi(2\alpha) + \dots + \pi(2n\alpha)$ and B is the derivative of P for $\theta = 0$. When ω is odd (resp. even), B (resp. A) is equal to 0, for instance $\omega = 0$ gives

$$\varphi\theta + \varphi(\theta + \alpha) + \varphi(\theta + 2\alpha) + \dots + \varphi(\theta + 2n\alpha) = B\varphi_1\theta$$

and $\omega = 1$ gives

$$\varphi\theta \cdot \varphi(\theta + \alpha) + \varphi(\theta + \alpha)\varphi(\theta + 2\alpha) + \dots + \varphi(\theta + 2n\alpha)\varphi\theta$$
$$= \varphi\alpha \cdot \varphi 2\alpha + \varphi 2\alpha \cdot \varphi 3\alpha + \dots + \varphi(2n - 1)\alpha \cdot \varphi 2n\alpha.$$

The second paragraph is not very explicit; Abel considers the functions

$$\psi\theta = \sum_{k=0}^{2n} \delta^{k\mu}\varphi(\theta + k\alpha),\ \psi_1\theta = \sum_{k=0}^{2n} \delta^{-k\mu}\varphi(\theta + k\alpha)$$

where δ is a primitive $(2n+1)$-th root of 1. Since $\psi(\theta+\alpha) = \delta^{2n\mu}\psi\theta$ and $\psi_1(\theta+\alpha) = \delta^{-2n\mu}\psi_1\theta$, the product $\psi\theta \cdot \psi_1\theta$ is invariant by $\theta \mapsto \theta + \alpha$. It is an even polynomial in the transformed elliptic integral $y = \varphi_1(a\theta)$, of the form $A(y^2 - f^2)$ where $f = \varphi_1\left(a\frac{m\varpi i}{2n+1}\right)$. Thus this product is 0 when $\theta = \frac{m\varpi i}{2n+1}$ and this gives a remarkable identity

$$0 = \varphi\left(\frac{m\varpi i}{2n+1}\right) + \delta^{\mu}\varphi\left(\frac{m\varpi i}{2n+1} + \alpha\right) + \delta^{2\mu}\varphi\left(\frac{m\varpi i}{2n+1} + 2\alpha\right) + \ldots$$
$$+\delta^{2n\mu}\varphi\left(\frac{m\varpi i}{2n+1} + 2n\alpha\right)$$

for a convenient m. Abel has announced this type of identity in the introduction of the *Précis d'une théorie des fonctions elliptiques*, published in 1829 (see our §8); Sylow and Kronecker have proposed proofs for them.

7 Development of the Theory of Transformation of Elliptic Functions

The theory of transformation and of complex multiplication was developed by Abel in the paper *Solution d'un problème général concernant la transformation des fonctions elliptiques* (*Astronomische Nachrichten* (6) 138 and (7) 147, 1828; *Œuvres*, t. I, p. 403–443), published in the Journal where Jacobi had announced the formulae for transformation. Abel deals with the following problem: "To find all the possible cases in which the differential equation

$$\frac{dy}{\sqrt{(1 - c_1^2 y^2)(1 - e_1^2 y^2)}} = \pm a \frac{dx}{\sqrt{(1 - c^2 x^2)(1 - e^2 x^2)}} \tag{93}$$

may be satisfied by putting for y an algebraic function of x, rational or irrational." He explains that the problem may be reduced to the case in which y is a rational function of x and he begins by solving this case. His notations are $x = \lambda\theta$ when $\theta = \int_0 \frac{dx}{\sqrt{(1-c^2x^2)(1-e^2x^2)}}$, $\Delta\theta = \sqrt{(1 - c^2x^2)(1 - e^2x^2)}$, $\frac{\omega}{2} = \int_0^{\frac{1}{c}} \frac{dx}{\sqrt{(1-c^2x^2)(1-e^2x^2)}}$, $\frac{\omega'}{2} = \int_0^{\frac{1}{e}} \frac{dx}{\sqrt{(1-c^2x^2)(1-e^2x^2)}}$ where e and c may be complex numbers. Abel recalls the addition theorem $\lambda(\theta \pm \theta') = \frac{\lambda\theta \cdot \Delta\theta' \pm \lambda\theta' \cdot \Delta\theta}{1 - c^2e^2\lambda^2\theta \cdot \lambda^2\theta'}$ and the solution of the equation $\lambda\theta' = \lambda\theta$, which is $\theta' = (-1)^{m+m'}\theta + m\omega + m'\omega'$. Let $y = \psi(x)$ be the rational function we are looking for and $x = \lambda\theta$, $x_1 = \lambda\theta_1$ two solutions of the equation $y = \psi(x)$, y being given (it is supposed that this equation is not of the first degree). From the equation $\frac{dy}{\sqrt{R}} = \pm a d\theta = \pm a d\theta_1$, we deduce $d\theta_1 = \pm d\theta$. Thus $\theta_1 = \alpha \pm \theta$ where α is constant and $x_1 = \lambda(\alpha \pm \theta)$, where we may choose the sign $+$, for $\lambda(\alpha - \theta) = \lambda(\omega - \alpha + \theta)$. Now $y = \psi(\lambda\theta) = \psi(\lambda(\theta + \alpha)) = \psi(\lambda(\theta + 2\alpha)) = \ldots = \psi(\lambda(\theta + k\alpha))$ for any integer k. As the equation $y = \psi(x)$ has only a finite number of roots, there exist k and k' distinct such that $\lambda(\theta + k\alpha) = \lambda(\theta + k'\alpha)$ or $\lambda(\theta + n\alpha) = \lambda\theta$ where $n = k - k'$ (supposed to be positive). Then $\theta + n\alpha = (-1)^{m+m'}\theta + m\omega + m'\omega'$ and, necessarily, $(-1)^{m+m'} = 1$, $n\alpha = m\omega + m'\omega'$ or $\alpha = \mu\omega + \mu'\omega'$ where μ, μ' are rational numbers. If the equation $y = \psi(x)$ has roots other than $\lambda(\theta + k\alpha)$, any one of them has the form $\lambda(\theta + \alpha_1)$ where $\alpha_1 = \mu_1\omega + \mu_1'\omega'$ (μ_1, μ_1' rational) and all the $\lambda(\theta + k\alpha + k_1\alpha_1)$ are roots of the equation. Continuing in this way, Abel finds that the roots of $y = \psi(x)$ are of the form

$$x = \lambda(\theta + k_1\alpha_1 + k_2\alpha_2 + \ldots + k_\nu\alpha_\nu)$$

where $k_1, k_2, \ldots, k_\nu$ are integers and $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ of the form $\mu\omega + \mu'\omega'$ (μ, μ' rational). The problem is to determine y in function of θ, the quantities $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ being given.

Before the solution of this problem, Abel deals with the case in which $y = \frac{f'+fx}{g'+gx}$. In this case $1 \pm c_1 y = \frac{g' \pm c_1 f' + (g \pm c_1 f)x}{g'+gx}$, $1 \pm e_1 y = \frac{g' \pm e_1 f' + (g \pm e_1 f)x}{g'+gx}$ and $dy = \frac{fg' - f'g}{(g'+gx)^2} dx$ so that the differential equation (93) takes the form

$$\begin{aligned} &\frac{fg' - f'g}{\sqrt{(g'^2 - c_1^2 f'^2)(g'^2 - e_1^2 f'^2)}} \\ &\times \frac{dx}{\sqrt{\left(1 + \frac{g+c_1 f}{g'+c_1 f'}x\right)\left(1 + \frac{g-c_1 f}{g'-c_1 f'}x\right)\left(1 + \frac{g+e_1 f}{g'+e_1 f'}x\right)\left(1 + \frac{g-e_1 f}{g'-e_1 f'}x\right)}} \\ &= \pm a \frac{dx}{\sqrt{(1 - c^2x^2)(1 - e^2x^2)}}. \end{aligned}$$

The solutions are $y = ax$, $c_1^2 = \frac{c^2}{a^2}$, $e_1^2 = \frac{e^2}{a^2}$; $y = \frac{a}{ec}\frac{1}{x}$, $c_1^2 = \frac{c^2}{a^2}$, $e_1^2 = \frac{e^2}{a^2}$; $y = m\frac{1 - x\sqrt{ec}}{1 + x\sqrt{ec}}$, $c_1 = \frac{1}{m}\frac{\sqrt{c} - \sqrt{e}}{\sqrt{c} + \sqrt{e}}$, $e_1 = \frac{1}{m}\frac{\sqrt{c} + \sqrt{e}}{\sqrt{c} - \sqrt{e}}$, $a = \frac{m\sqrt{-1}}{2}(c - e)$.

In order to deal with the general case, in which the solutions of $y = \psi(x)$ are

$$\lambda\theta, \lambda(\theta + \alpha_1), \ldots, \lambda(\theta + \alpha_{m-1}),$$

Abel writes $\psi(x) = \frac{p}{q}$ where p and q are polynomials of degree m in x, with respective dominant coefficients f and g. The equation $y = \psi(x)$ is rewritten

$$p - qy = (f - gy)(x - \lambda\theta)(x - \lambda(\theta + \alpha_1)) \ldots (x - \lambda(\theta + \alpha_{m-1})). \tag{94}$$

If f' and g' are the respective coefficients of x^{m-1} in p and q, we see that

$$f' - g'y = -(f - gy)(\lambda\theta + \lambda(\theta + \alpha_1) + \ldots + \lambda(\theta + \alpha_{m-1}))$$

and $y = \frac{f'+f\cdot\varphi\theta}{g'+g\cdot\varphi\theta}$ where $\varphi\theta = \lambda\theta + \lambda(\theta + \alpha_1) + \ldots + \lambda(\theta + \alpha_{m-1})$. It remains to express $\varphi\theta$ rationally in function of x with the help of the addition theorem and to determine f, f', g, g', e_1, c_1 and a in order that (93) be satisfied. For some α_j, it is possible that $\lambda(\theta-\alpha_j) = \lambda(\theta+\alpha_j)$ or $\lambda(\theta+2\alpha_j) = \lambda\theta$. Then $\alpha_j = \frac{m}{2}\omega + \frac{m'}{2}\omega'$ with $m+m'$ even and the distinct values of $\lambda(\theta+\alpha_j)$ are $\lambda\theta = x$, $\lambda(\theta+\omega) = -\lambda\theta = -x$, $\lambda\left(\theta + \frac{\omega}{2} + \frac{\omega'}{2}\right) = -\frac{1}{ec}\frac{1}{\lambda\theta} = -\frac{1}{ec}\frac{1}{x}$, $\lambda\left(\theta + \frac{3\omega}{2} + \frac{\omega'}{2}\right) = -\frac{1}{ec}\frac{1}{\lambda(\theta+\omega)} = \frac{1}{ec}\frac{1}{x}$. For the other $\lambda(\theta - \alpha_j) \neq \lambda(\theta + \alpha_j)$, so it is a root of the equation $y = \psi(x)$, of the form $\lambda(\theta + \alpha_{j'})$ and we have $\lambda(\theta + \alpha_j) + \lambda(\theta - \alpha_j) = \frac{2x\Delta\alpha_j}{1-e^2c^2\lambda^2\alpha_j x^2}$. Thus

$$\begin{aligned}\varphi\theta &= \lambda\theta + k\lambda(\theta+\omega) + k'\lambda\left(\theta + \frac{\omega}{2} + \frac{\omega'}{2}\right) + k''\lambda\left(\theta + \frac{3\omega}{2} + \frac{\omega'}{2}\right)\\ &\quad + \lambda(\theta+\alpha_1) + \lambda(\theta-\alpha_1) + \ldots + \lambda(\theta+\alpha_n) + \lambda(\theta-\alpha_n)\\ &= (1-k)x + \frac{k''-k'}{ec}\frac{1}{x} + \sum \frac{2x\Delta\alpha_j}{1-e^2c^2\lambda^2\alpha_j\cdot x^2}\end{aligned}$$

where k, k', k'' are equal to 0 or 1.

In the first case considered by Abel, $k = k' = k'' = 0$. Let $\delta, \delta', \varepsilon, \varepsilon'$ be the values of θ respectively corresponding to $y = \frac{1}{c_1}, -\frac{1}{c_1}, \frac{1}{e_1}, -\frac{1}{e_1}$. One has $1 - c_1 y = \frac{g'-c_1 f'}{r}\left(1 - \frac{\varphi\theta}{\varphi\delta}\right)$, $1 + c_1 y = \frac{g'+c_1 f'}{r}\left(1 - \frac{\varphi\theta}{\varphi\delta'}\right)$, $1 - e_1 y = \frac{g'-e_1 f'}{r}\left(1 - \frac{\varphi\theta}{\varphi\varepsilon}\right)$ and $1 + e_1 y = \frac{g'+e_1 f'}{r}\left(1 - \frac{\varphi\theta}{\varphi\varepsilon'}\right)$ where $r = g' + g\cdot\varphi\theta$. From the expression of $\varphi\theta$, one gets $1 - \frac{\varphi\theta}{\varphi\delta} = \frac{1+A_1x+A_2x^2+\ldots+A_{2n+1}x^{2n+1}}{(1-e^2c^2\lambda^2\alpha_1\cdot x^2)(1-e^2c^2\lambda^2\alpha_2\cdot x^2)\ldots(1-e^2c^2\lambda^2\alpha_n\cdot x^2)}$, which must be annihilated by $\theta = \delta, \delta \pm \alpha_1, \ldots, \delta \pm \alpha_n$ (δ arbitrary). Thus

$$\begin{aligned}1 + A_1x + \ldots + A_{2n+1}x^{2n+1} &= \left(1 - \frac{x}{\lambda\delta}\right)\left(1 - \frac{x}{\lambda(\delta+\alpha_1)}\right)\left(1 - \frac{x}{\lambda(\delta-\alpha_1)}\right)\ldots\\ &\quad\times\left(1 - \frac{x}{\lambda(\delta+\alpha_n)}\right)\left(1 - \frac{x}{\lambda(\delta-\alpha_n)}\right).\end{aligned}$$

The differential equation (93) is written

$$\sqrt{(1-c_1^2y^2)(1-e_1^2y^2)} = \frac{1}{a}\frac{dy}{dx}\sqrt{(1-c^2x^2)(1-e^2x^2)}$$

and it shows that when $x = \pm\frac{1}{c}, \pm\frac{1}{e}$ or $\theta = \pm\frac{\omega}{2}, \pm\frac{\omega'}{2}$, the left hand side is 0. Thus, for instance, $\delta = \frac{\omega}{2}$, $\delta' = -\frac{\omega}{2}$, $\varepsilon = \frac{\omega'}{2}$, $\varepsilon' = -\frac{\omega'}{2}$ and $g' = c_1 f\varphi\left(\frac{\omega}{2}\right) = e_1 f\varphi\left(\frac{\omega'}{2}\right)$, $f' = \frac{g}{c_1}\varphi\left(\frac{\omega}{2}\right) = \frac{g}{e_1}\varphi\left(\frac{\omega'}{2}\right)$. A solution of this system is $g = f' = 0$, $\frac{f}{g'} = \frac{1}{k}$, $c_1 = \frac{k}{\varphi(\frac{\omega}{2})}$, $e_1 = \frac{k}{\varphi\left(\frac{\omega'}{2}\right)}$ where k is arbitrary. Then $y = \frac{1}{k}\varphi\theta$ and $1 - \frac{\varphi\theta}{\varphi(\frac{\omega}{2})} = \frac{1}{\rho}(1-cx)\left(1 - \frac{x}{\lambda(\frac{\omega}{2}-\alpha_1)}\right)^2\left(1 - \frac{x}{\lambda(\frac{\omega}{2}-\alpha_2)}\right)^2\ldots\left(1 - \frac{x}{\lambda(\frac{\omega}{2}-\alpha_n)}\right)^2$ where

$$\rho = (1 - e^2c^2\lambda^2\alpha_1 \cdot x^2)(1 - e^2c^2\lambda^2\alpha_2 \cdot x^2)\dots(1 - e^2c^2\lambda^2\alpha_n \cdot x^2).$$

We obtain similar expressions for $1+\frac{\varphi\theta}{\varphi\left(\frac{\omega}{2}\right)}$ and for $1\pm\frac{\varphi\theta}{\varphi\left(\frac{\omega'}{2}\right)}$ and, as in the *Recherches*,

$$1 - c_1^2y^2 = (1 - c^2x^2)\frac{t^2}{\rho^2}, \quad 1 - e_1^2y^2 = (1 - e^2x^2)\frac{t'^2}{\rho^2}$$

where

$$t = \left(1 - \frac{x^2}{\lambda^2\left(\frac{\omega}{2} - \alpha_1\right)}\right)\left(1 - \frac{x^2}{\lambda^2\left(\frac{\omega}{2} - \alpha_2\right)}\right)\dots\left(1 - \frac{x^2}{\lambda^2\left(\frac{\omega}{2} - \alpha_n\right)}\right),$$

$$t' = \left(1 - \frac{x^2}{\lambda^2\left(\frac{\omega'}{2} - \alpha_1\right)}\right)\left(1 - \frac{x^2}{\lambda^2\left(\frac{\omega'}{2} - \alpha_2\right)}\right)\dots\left(1 - \frac{x^2}{\lambda^2\left(\frac{\omega'}{2} - \alpha_n\right)}\right).$$

Thus $\sqrt{(1 - c_1^2y^2)(1 - e_1^2y^2)} = \pm\frac{tt'}{\rho^2}\sqrt{(1 - c^2x^2)(1 - e^2x^2)}$ and Abel shows, as in the *Recherches*, that $\frac{\rho^2\frac{dy}{dx}}{tt'}$ is a constant a, so that the desired result is obtained. The value of a is computed by comparing the limit values of $\frac{dy}{dx}$ for x infinite coming from $\frac{dy}{dx} = a\frac{tt'}{\rho^2}$ and from $y = \frac{1}{k}\left(x + 2x\sum\frac{\Delta(\alpha)}{1-e^2c^2\lambda^2\alpha\cdot x^2}\right)$. Abel finds $a = (e^2c^2)^n\frac{1}{k}\lambda^4\alpha_1 \cdot \lambda^4\alpha_2 \dots \lambda^4\alpha_n$. He gives some other forms for y, as

$$\begin{aligned} y &= a\frac{x\left(1 - \frac{x^2}{\lambda^2\alpha_1}\right)\left(1 - \frac{x^2}{\lambda^2\alpha_2}\right)\dots\left(1 - \frac{x^2}{\lambda^2\alpha_n}\right)}{(1 - e^2c^2\lambda^2\alpha_1 \cdot x^2)(1 - e^2c^2\lambda^2\alpha_2 \cdot x^2)\dots(1 - e^2c^2\lambda^2\alpha_n \cdot x^2)} \\ &= \frac{1}{k}(ec)^{2n}b\lambda\theta \cdot \lambda(\alpha_1 + \theta)\lambda(\alpha_1 - \theta)\dots\lambda(\alpha_n + \theta)\lambda(\alpha_n - \theta) \end{aligned}$$

where $b = \lambda^2\alpha_1 \cdot \lambda^2\alpha_2 \dots \lambda^2\alpha_n$. Doing $\theta = \frac{\omega}{2}$ and $\theta = \frac{\omega'}{2}$, he obtains

$$\frac{1}{c_1} = (-1)^n\frac{b}{k}e^{2n}c^{2n-1}\left(\lambda\left(\frac{\omega}{2} - \alpha_1\right)\lambda\left(\frac{\omega}{2} - \alpha_2\right)\dots\lambda\left(\frac{\omega}{2} - \alpha_n\right)\right)^2 \text{ and}$$

$$\frac{1}{e_1} = (-1)^n\frac{b}{k}e^{2n-1}c^{2n}\left(\lambda\left(\frac{\omega'}{2} - \alpha_1\right)\lambda\left(\frac{\omega'}{2} - \alpha_2\right)\dots\lambda\left(\frac{\omega'}{2} - \alpha_n\right)\right)^2.$$

As Abel remarks, the transformation defined by Jacobi corresponds to the case in which $\alpha_1 = \frac{2\omega}{2n+1}$, $c = c_1 = 1$ and the theory explained in the *Recherches* to the case in which

$$\alpha_1 = \frac{m\omega + m'\omega'}{2n + 1}$$

with $m+m'$ even, m, m' and $2n+1$ having no common factor. In both cases, $\alpha_2 = 2\alpha_1$, $\alpha_3 = 3\alpha_1, \dots, \alpha_n = n\alpha_n$. Jacobi independently found these transformations. The more general transformation $y = \frac{f' + f \cdot \varphi\theta}{g' + g \cdot \varphi\theta}$ is obtained by composing this particular one with a transformation of the type $y = \frac{f' + fx}{g' + gx}$.

A second case considered by Abel is that in which $k = 0$ and k' or k'' is equal to 1. It is impossible that $k' = k'' = 1$ for if $\lambda\left(\theta + \frac{\omega}{2} + \frac{\omega'}{2}\right)$ and $\lambda\left(\theta + \frac{3\omega}{2} + \frac{\omega'}{2}\right)$ are roots of $y = \psi(x)$, so is $\lambda\left(\theta + \frac{3\omega}{2} + \frac{\omega'}{2} - \frac{\omega+\omega'}{2}\right) = \lambda(\theta + \omega)$ and k is not 0. As in the first case, let $1 - c_1 y = 0$ for $x = \frac{1}{c}$. Then $1 \pm c_1 y = \frac{g' \pm c_1 f'}{r}\left(1 - \frac{\varphi\theta}{\varphi\left(\frac{\omega}{2}\right)}\right)$, $1 - c_1^2 y^2 = \frac{g'^2 - c_1^2 f'^2}{r^2}\left(1 - \left(\frac{\varphi\theta}{\varphi\left(\frac{\omega}{2}\right)}\right)^2\right)$ and

$$1 - \frac{\varphi\theta}{\varphi\delta} = -\frac{1}{\varphi\delta \cdot \rho}\left(1 - \frac{x}{\lambda\delta}\right)\left(1 - \frac{x}{\lambda(\delta+\beta)}\right)\left(1 - \frac{x}{\lambda(\delta+\alpha_1)}\right)\left(1 - \frac{x}{\lambda(\delta-\alpha_1)}\right)$$
$$\times \cdots \left(1 - \frac{x}{\lambda(\delta+\alpha_n)}\right)\left(1 - \frac{x}{\lambda(\delta-\alpha_n)}\right) \tag{95}$$

where $\beta = \frac{\omega+\omega'}{2}$ (resp. $\frac{3\omega+\omega'}{2}$) if $k' = 1$ (resp. $k'' = 1$) and

$$\rho = \pm ecx(1 - e^2c^2\lambda^2\alpha_1 \cdot x^2)(1 - e^2c^2\lambda^2\alpha_2 \cdot x^2)\ldots(1 - e^2c^2\lambda^2\alpha_n \cdot x^2).$$

Abel takes $\delta = \pm\frac{\omega}{2}$ in order to compute $1 - c_1^2 y^2$ and he finds

$$\sqrt{1 - c_1^2 y^2} = \frac{\sqrt{c_1^2 f'^2 - g'^2}}{\varphi\left(\frac{\omega}{2}\right) r\rho} t\sqrt{(1 - c^2x^2)(1 - e^2x^2)}.$$

Now it results from (93) that $\sqrt{1 - e_1^2 y^2} = \frac{\varphi\left(\frac{\omega}{2}\right)}{a\sqrt{c_1^2 f'^2 - g'^2}} \frac{r\rho}{t} \frac{dy}{dx}$ is a rational function of x. If we impose that $1 - e_1^2 y^2$ be annihilated by $x = \pm\lambda\left(\frac{\omega-\beta}{2}\right)$, we effectively find

$$\sqrt{1 - e_1^2 y^2} = \frac{\sqrt{e_1^2 f'^2 - g'^2}}{\varphi\left(\frac{\omega-\beta}{2}\right) r\rho}\left(1 - \frac{x^2}{\lambda^2\left(\frac{\omega-\beta}{2}\right)}\right)\left(1 - \frac{x^2}{\lambda^2\left(\frac{\omega-\beta}{2} - \alpha_1\right)}\right)\cdots$$
$$\times\left(1 - \frac{x^2}{\lambda^2\left(\frac{\omega-\beta}{2} - \alpha_n\right)}\right)$$

by doing $\delta = \pm\frac{\omega-\beta}{2}$ in the relation (95). Then we find that $g' = c_1 f \cdot \varphi\left(\frac{\omega}{2}\right) = e_1 f \cdot \varphi\left(\frac{\omega-\beta}{2}\right)$, $f' = \frac{g}{c_1}\varphi\left(\frac{\omega}{2}\right) = \frac{g}{e_1}\varphi\left(\frac{\omega-\beta}{2}\right)$. A solution is $f = g' = 0$, $\frac{f}{g'} = \frac{\varphi\left(\frac{\omega}{2}\right)}{c_1} = \frac{\varphi\left(\frac{\omega-\beta}{2}\right)}{e_1}$, which gives $c_1 = k\varphi\left(\frac{\omega}{2}\right)$, $e_1 = k\varphi\left(\frac{\omega-\beta}{2}\right)$, $y = \frac{1}{k\varphi\theta}$, $a = \pm\frac{ec}{k}$. Other solutions are obtained by composing with a transformation of the type $y = \frac{f' + fx}{g' + gx}$.

In the simplest case where $n = 0$ and $c_1 = c = 1$, $\beta = \frac{3\omega}{2} + \frac{\omega'}{2}$, the formulae are $y = (1+e)\frac{x}{1+ex^2}$, $e_1 = \frac{2\sqrt{e}}{1+e}$ and $a = 1+e$.

In the third case, $k = 1$ and one finds that $\varphi\theta = \varphi(\theta+\omega) = -\varphi\theta$ so that $\varphi\theta = 0$. Let us return to (94), denoting by $\frac{1}{2}f'$ (resp. $\frac{1}{2}g'$) the coefficient of x^{m-2} in p (resp. q) and by $F\theta$ the function $\lambda^2\theta + \lambda^2(\theta+\alpha_1) + \ldots + \lambda^2(\theta+\alpha_{m-1})$. We have $f' - g'y = -(f-gy)F\theta$ and $y = \frac{f'+f\cdot F\theta}{g'+g\cdot F\theta}$ and we may proceed as in the preceding cases with $F\theta$ in place of $\varphi\theta$.

Abel states the general theorem for the first real transformation of arbitrary order n:

$$\frac{dy}{\sqrt{(1-y^2)(1-e_1^2y^2)}} = \pm\frac{adx}{\sqrt{(1-x^2)(1-e^2x^2)}} \tag{96}$$

where $a = k\lambda\frac{\omega}{n}\cdot\lambda\frac{2\omega}{n}\ldots\lambda\frac{(n-1)\omega}{n}$, $e_1 = e^n\left(\lambda\frac{\omega}{2n}\cdot\lambda\frac{3\omega}{2n}\ldots\lambda\left(n-\frac{1}{2}\right)\frac{\omega}{n}\right)^2$, $1 = k\lambda\frac{\omega}{2n}\cdot\lambda\frac{3\omega}{2n}\times\ldots\lambda\left(n-\frac{1}{2}\right)\frac{\omega}{n}$ and $y = k\lambda\theta\cdot\lambda\left(\theta+\frac{\omega}{n}\right)\lambda\left(\theta+\frac{2\omega}{n}\right)\ldots\lambda\left(\theta+\frac{(n-1)\omega}{n}\right)$.

With Legendre's notation $x = \sin\varphi$, $y = \sin\psi$ and n very large, e_1 becomes negligible and Abel writes, with an approximation $\psi = a\int\limits_0^\varphi \frac{d\varphi}{\sqrt{1-e^2\sin^2\varphi}} = \sum\limits_{m=0}^{n-1}\arctan\left(\tan\varphi\sqrt{1-e^2\lambda^2\left(\frac{m\omega}{n}\right)}\right)$. For $\varphi = \frac{\pi}{2}$, $\psi = n\frac{\pi}{2} = a\frac{\omega}{2}$, so that $\frac{1}{a} = \frac{1}{\pi}\frac{\omega}{n}$ and passing to the limit for n infinite, Abel finds

$$\int\limits_0^\varphi \frac{d\varphi}{\sqrt{1-e^2\sin^2\varphi}} = \frac{1}{\pi}\int\limits_0^\omega \arctan\left(\tan\varphi\sqrt{1-e^2\lambda^2 x}\right)dx.$$

The order of the transformation, that is the degree m of the equation $p - qy = 0$ is the number of distinct values of $\lambda(\theta + k_1\alpha_1 + \ldots + k_\nu\alpha_\nu)$ and Abel shows that $m = n_1n_2\ldots n_\nu$ where, for each j, n_j is the smallest strictly positive integer such that $\lambda(\theta+n_j\alpha_j) = \lambda(\theta+m_1\alpha_1+\ldots+m_{j-1}\alpha_{j-1})$ for some $m_1,\ldots,m_{j-1}$. Thus, when m is a prime number, $\nu = 1$ and $m = n_1$. Abel states some theorems:

a) when the order of a transformation is a composite number mn, this transformation may be obtained by the composition of a transformation of order m and a transformation of order n;

b) the equation $y = \psi(x)$ is algebraically solvable and its roots x are rational functions of y and some radicals $r_1^{\frac{1}{n_1}}, r_2^{\frac{1}{n_2}},\ldots,r_\nu^{\frac{1}{n_\nu}}$ where $n_1, n_2, \ldots, n_\nu$ are prime numbers, $n_1n_2\cdots n_\nu$ is the degree of the equation and $r_1, r_2, \ldots, r_\nu$ have the form $\zeta + t\sqrt{(1-c_1^2y^2)(1-e_1^2y^2)}$ with ζ and t rational in y;

c) If the differential equation $\frac{dy}{\sqrt{(1-c^2y^2)(1-e^2y^2)}} = a\frac{dx}{\sqrt{(1-c^2x^2)(1-e^2x^2)}}$ has a solution algebraic in x and y, $a = \mu' + \sqrt{-\mu}$ where μ', μ are rational numbers and $\mu \geq 0$. There is an infinity of values for the moduli e, c, given by algebraic equations solvable by radicals, for which $\mu > 0$.

Recall that in the *Recherches* Abel doubted that these equations might be algebraically solvable. Kronecker (1857) gave a proof that they are solvable by radicals, as Abel states here.

d) If the differential equation $\frac{dy}{\sqrt{(1-y^2)(1-b^2y^2)}} = a\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$ where $b^2 = 1-c^2$ has a solution algebraic in x and y, $a = \sqrt{\mu + \mu'\sqrt{-1}}$ where μ, μ' are rational numbers and $\mu \geq 0$. In particular, when a is real, it is the square root of a positive rational number. Thus, when $e_1^2 = 1 - e^2$ in (96), $a = \sqrt{n}$. Indeed the formula for k gives $y = 1$ when $\theta = \frac{\omega}{2n}$, thus

$$\int\limits_0^1 \frac{dy}{\sqrt{(1-y^2)(1-e_1^2y^2)}} = \frac{a\omega}{2n}.$$

One may write $y = k\lambda\theta \cdot \lambda\left(\frac{\omega}{n} - \theta\right)\lambda\left(\frac{2\omega}{n} - \theta\right)\ldots\lambda\left(\frac{(n-1)\omega}{n} - \theta\right)$ because $\lambda\left(\theta + \frac{m\omega}{n}\right) = \lambda\left(\frac{(n-m)\omega}{n} - \theta\right)$, so that $y^2 = k^2x^2\frac{\lambda^2\frac{\omega}{n} - x^2}{1-e^2\lambda^2\frac{\omega}{n}x^2}\cdots\frac{\lambda^2\frac{(n-1)\omega}{n} - x^2}{1-e^2\lambda^2\frac{(n-1)\omega}{n}x^2}$. Now putting $x = p\sqrt{-1}$ and $y = z\sqrt{-1}$ and letting p and z tend towards infinity, Abel obtains $\frac{\omega}{2} = a\int\limits_0^1 \frac{dy}{\sqrt{(1-y^2)(1-e_1^2y^2)}} = a\frac{a\omega}{2n}$, whence $a = \sqrt{n}$.

In the part of this memoir published in 1829, Abel gives another study of the same transformation in the case in which $0 < c, c_1 < 1$ and $e = e_1 = 1$. Then $\frac{\omega}{2} = \int\limits_0^1 \frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$ is real but $\frac{\omega'}{2} = \int\limits_0^{\frac{1}{c}} \frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}} = \frac{\omega}{2}\sqrt{-1}\int\limits_1^{\frac{1}{c}} \frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}} = \frac{\omega}{2}\sqrt{-1}\frac{\varpi}{2}$, where $\frac{\varpi}{2} = \int\limits_0^1 \frac{dx}{\sqrt{(1-x^2)(1-b^2x^2)}}$, $b = \sqrt{1-c^2}$, is complex. Let us suppose that the differential equation

$$\frac{dy}{\sqrt{(1-y^2)(1-c_1^2y^2)}} = a\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}} \tag{97}$$

has a solution $f(y, x) = 0$ algebraic in x and y and define the function $y = \lambda_1\theta'$ by

$$\frac{dy}{\sqrt{(1-y^2)(1-c_1^2y^2)}} = d\theta' \quad \text{and} \quad \lambda_1(0) = 0.$$

The equation (97) takes the form $d\theta' = ad\theta$, so that $\theta' = \varepsilon + a\theta$ where ε is constant and $y = \lambda_1(\varepsilon + a\theta)$. Thus $f(\lambda_1(\varepsilon + a\theta), \lambda\theta) = 0$ identically in θ. This implies that $f(\lambda_1(\varepsilon + 2ma\omega + a\theta), \lambda\theta) = f(\lambda_1(\varepsilon + ma\varpi i + a\theta)\lambda\theta)$ for any integer m. Then there exists pairs of distinct integers (k, k') and (ν, ν') such that

$$\lambda_1(\varepsilon + 2k'a\omega + a\theta) = \lambda_1(\varepsilon + 2ka\omega + a\theta) \quad \text{and}$$
$$\lambda_1(\varepsilon + \nu'a\varpi i + a\theta) = \lambda_1(\varepsilon + \nu a\varpi i + a\theta)$$

or $2k'a\omega = 2ka\omega + 2m\omega_1 + m'\varpi\sqrt{-1}$, $\nu' a\varpi i = \nu a\varpi i + 2\mu\omega_1 + \mu'\varpi\sqrt{-1}$ where m, m', μ, μ' are integers, $\frac{\omega_1}{2} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-c_1^2x^2)}}$, $\frac{\varpi_1}{2}\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-b_1^2x^2)}}$, $b_1 = \sqrt{1-c_1^2}$. From these relations we draw $a = \frac{m}{\nu}\frac{\omega_1}{\omega} + \frac{m'}{2\nu}\frac{\varpi_1}{\omega}\sqrt{-1} = \frac{\mu'}{\nu'}\frac{\varpi_1}{\varpi} - \frac{2\mu}{\nu'}\frac{\omega_1}{\varpi}\sqrt{-1}$ and $\frac{m}{\nu}\frac{\omega_1}{\omega} = \frac{\mu'}{\nu'}\frac{\varpi_1}{\varpi}$, $\frac{m'}{2\nu}\frac{\varpi_1}{\omega} = -\frac{2\mu}{\nu'}\frac{\omega_1}{\varpi}$. Thus $\frac{\omega^2}{\varpi^2} = -\frac{1}{4}\frac{mm'}{\mu\mu'}\frac{\nu'^2}{\nu^2}$ and $\frac{\omega_1^2}{\varpi_1^2} = -\frac{1}{4}\frac{m'\mu'}{m\mu}$. As $\frac{\omega^2}{\varpi^2}$ is a continuous function of c, these equations can be satisfied for any c, c_1 only if $m' = \mu = 0$ or if $m = \mu' = 0$. In the first case we have $a = \frac{m}{\nu}\frac{\omega_1}{\omega} = \frac{\mu'}{\nu'}\frac{\varpi_1}{\varpi}$, $\frac{\omega_1}{\varpi_1} = \frac{\nu\mu'}{\nu' m}\frac{\omega}{\varpi}$ and in the second case $a = \frac{m'}{2\nu}\frac{\varpi_1}{\omega}\sqrt{-1} = -\frac{2\mu}{\nu'}\frac{\omega_1}{\varpi}\sqrt{-1}$, $\frac{\omega_1}{\varpi_1} = -\frac{1}{4}\frac{m'\nu'}{\mu\nu}\frac{\varpi}{\omega}$. Abel states that if (97) has a solution algebraic in x and y, then either $\frac{\omega_1}{\varpi_1}$ or $\frac{\varpi_1}{\omega_1}$ has a rational ratio to $\frac{\omega}{\varpi}$. In the first case $a = \delta\frac{\omega_1}{\omega}$ and in the second case $a = \delta\frac{\varpi_1}{\omega}\sqrt{-1}$, with δ rational. Both ratios k, k' are rational for certain particular values of c, c_1, determined by $\frac{\varpi}{\omega} = \sqrt{kk'}$, $\frac{\varpi_1}{\omega_1} = \sqrt{\frac{k'}{k}}$ and in these cases $a = \delta\frac{\omega_1}{\omega} + \delta'\frac{\varpi_1}{\omega}\sqrt{-1}$ with δ, δ' rational.

In order to prove that these conditions are sufficient for the existence of an algebraic solution to the equation (97), Abel observes that $\lambda\alpha = f\left(b\frac{\omega}{2} - b\alpha\right)$ where $f\alpha = \sqrt{1-x^2}$ is the function introduced in the *Recherches*. The expansion of $f\alpha$ in simple infinite product then gives

$$\begin{aligned}\lambda\alpha &= A\frac{(1-t^2)(1-t^2r^2)(1-t^{-2}r^2)(1-t^2r^4)(1-t^{-2}r^4)\dots}{(1+t^2)(1+t^2r^2)(1+t^{-2}r^2)(1+t^2r^4)(1+t^{-2}r^4)\dots}\\ &= A\psi\left(\alpha\frac{\pi}{\varpi}\right)\psi(\omega+\alpha)\frac{\pi}{\varpi}\psi(\omega-\alpha)\frac{\pi}{\varpi}\psi(2\omega+\alpha)\frac{\pi}{\varpi}\psi(2\omega-\alpha)\frac{\pi}{\varpi}\dots \qquad (98)\end{aligned}$$

where A is independent from α, $t = e^{-\frac{\alpha\pi}{\varpi}}$, $r = e^{-\frac{\omega}{\varpi}\pi}$ and $\psi(x) = \frac{1-e^{-2x}}{1+e^{-2x}}$. From this formula we draw

$$\begin{aligned}&\lambda\theta\cdot\lambda\left(\theta+\frac{\omega}{n}\right)\lambda\left(\theta+\frac{2\omega}{n}\right)\dots\lambda\left(\theta+\frac{n-1}{n}\omega\right)\\ &= A^n\psi\delta\frac{\pi}{\varpi_1}\cdot\psi(\omega_1+\delta)\frac{\pi}{\varpi_1}\psi(\omega_1-\delta)\frac{\pi}{\varpi_1}\psi(2\omega_1+\delta)\frac{\pi}{\varpi_1}\dots\end{aligned}$$

where $\delta = \frac{\varpi_1}{\varpi}\theta$ and $\frac{\omega_1}{\varpi_1} = \frac{1}{n}\frac{\omega}{\varpi}$. On the other hand

$$\lambda_1\alpha = A_1\psi\left(\alpha\frac{\pi}{\varpi_1}\right)\psi(\omega_1+\alpha)\frac{\pi}{\varpi_1}\cdot\psi(\omega_1-\alpha)\frac{\pi}{\varpi_1}\dots$$

and, by comparison, we conclude that $\lambda_1\left(\frac{\varpi_1}{\varpi}\theta\right) = \frac{A_1}{A^n}\lambda\theta\cdot\lambda\left(\theta+\frac{\omega}{n}\right)\lambda\left(\theta+\frac{2\omega}{n}\right)$ $\times\dots\lambda\left(\theta+\frac{n-1}{n}\omega\right) = y$, algebraic function of $x = \lambda\theta$ such that $\frac{dy}{\sqrt{(1-y^2)(1-c_1^2y^2)}} = \frac{\varpi_1}{\varpi}d\theta = \frac{\varpi_1}{\varpi}\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$.

There are three cases to consider: a real, a purely imaginary and a complex with $\frac{\varpi}{\omega} = \sqrt{kk'}$ and $\frac{\varpi_1}{\omega_1} = \sqrt{\frac{k'}{k}}$. In the first case, $a = \frac{\mu}{\nu}\frac{\varpi_1}{\varpi}$ and $\frac{\omega_1}{\varpi_1} = \frac{m}{n}\frac{\omega}{\varpi}$ where μ, ν, m, n

are integers. The equation (97) is satisfied by $x = \lambda(\nu\varpi\theta)$, $y = \lambda_1(\mu\varpi_1\theta)$. Let c' be a modulus such that $\frac{\omega'}{\varpi'} = \frac{1}{n}\frac{\omega}{\varpi}$, where ω' and ϖ' are the periods corresponding to c'. We have

$$\lambda'(\mu\nu\varpi'\theta) = \frac{A'}{A^n}\lambda(\mu\nu\varpi\theta)\lambda\left(\mu\nu\varpi\theta + \frac{\omega}{n}\right)\ldots\lambda\left(\mu\nu\varpi\theta + \frac{n-1}{n}\omega\right)$$

and since $\frac{\omega'}{\varpi'} = \frac{1}{m}\frac{\omega_1}{\varpi_1}$,

$$\lambda'(\mu\nu\varpi'\theta) = \frac{A'}{A_1^m}\lambda_1(\mu\nu\varpi_1\theta)\lambda_1\left(\mu\nu\varpi_1\theta + \frac{\omega_1}{m}\right)\cdots\lambda_1\left(\mu\nu\varpi_1\theta + \frac{m-1}{m}\omega_1\right).$$

Finally

$$\begin{aligned}&\frac{1}{A^n}\lambda(\mu\delta)\lambda\left(\mu\delta + \frac{\omega}{n}\right)\cdots\lambda\left(\mu\delta + \frac{n-1}{n}\omega\right)\\&= \frac{1}{A_1^m}\lambda_1(\nu\delta_1)\lambda_1\left(\nu\delta_1 + \frac{\omega_1}{m}\right)\cdots\lambda_1\left(\nu\delta_1 + \frac{m-1}{m}\omega_1\right)\end{aligned} \tag{99}$$

where $\nu\varpi\theta = \delta$ and $\mu\varpi_1\theta = \delta_1$. The left hand side is an algebraic function of $\lambda(\mu\delta)$, so an algebraic function of $x = \lambda\delta$ and, in the same way, the right hand side is an algebraic function of $y = \lambda_1\delta_1$. Thus we have an algebraic integral of (97). One sees that $A = \frac{1}{\sqrt{c}}$ and $A_1 = \frac{1}{\sqrt{c_1}}$. As an example, Abel explains the case in which $a = \frac{\varpi_1}{\varpi}$ and $\frac{\omega_1}{\varpi_1} = \frac{2}{3}\frac{\omega}{\varpi}$; the equation (99) takes the form $c\sqrt{c}\cdot\lambda\left(\delta + \frac{\omega}{3}\right)\lambda\left(\delta + \frac{2\omega}{3}\right) = c_1\lambda_1\delta_1\cdot\lambda_1\left(\delta_1 + \frac{\omega_1}{2}\right)$ or $y\frac{\sqrt{1-y^2}}{\sqrt{1-c_1^2y^2}} = \frac{c\sqrt{c}}{c_1}x\frac{\lambda^2\frac{\omega}{3}-x^2}{1-c^2\lambda^2\frac{\omega}{3}x^2}$.

In the second case $a = \frac{\mu}{\nu}\frac{\varpi_1}{\omega}\sqrt{-1}$ and $\frac{\omega_1}{\varpi_1} = \frac{m}{n}\frac{\varpi}{\omega}$ with μ, ν, m, n integers. Let us put $x = \frac{z\sqrt{-1}}{\sqrt{1-z^2}}$ so that $\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}} = \sqrt{-1}\frac{dz}{\sqrt{(1-z^2)(1-b^2z^2)}}$ where $b = \sqrt{1-c^2}$. The equation (97) takes the form $\frac{dy}{\sqrt{(1-y^2)(1-c_1^2y^2)}} = \frac{\mu}{\nu}\frac{\varpi_1}{\omega}\frac{dz}{\sqrt{(1-z^2)(1-b^2z^2)}}$ and we are reduced to the preceding case, with the algebraic integral (99) where $z = \lambda\delta = \frac{x}{\sqrt{x^2-1}}$ and ω replaced by ϖ. For instance if $a = \frac{\varpi_1}{\omega}\sqrt{-1}$ and $\frac{\omega_1}{\varpi_1} = 2\frac{\varpi}{\omega}$, (99) is written $\sqrt{b}\cdot\lambda\delta = c_1\lambda_1(\delta_1)\lambda_1\left(\delta_1 + \frac{\omega_1}{2}\right)$ or $y\frac{\sqrt{1-y^2}}{\sqrt{1-c_1^2y^2}} = \frac{\sqrt{b}}{c_1}\frac{x}{\sqrt{x^2-1}}$.

In the third case $a = \frac{\mu}{\nu}\frac{\varpi_1}{\varpi} + \frac{\mu'}{\nu'}\frac{\varpi_1}{\omega}\sqrt{-1}$ where μ, ν, μ', ν' are integers and $\frac{\omega_1}{\varpi_1} = k\frac{\omega}{\varpi} = \frac{1}{k'}\frac{\varpi}{\omega}$ where k, k' are rational numbers. The two equations

$$\frac{dz}{\sqrt{(1-z^2)(1-c_1^2z^2)}} = \frac{\mu}{\nu}\frac{\varpi_1}{\omega}\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}} \quad \text{and}$$

$$\frac{dv}{\sqrt{(1-v^2)(1-c_1^2v^2)}} = \frac{\mu'}{\nu'}\frac{\varpi_1}{\omega}\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$$

have algebraic integrals and our equation (97) may be written

$$\frac{dy}{\sqrt{(1-y^2)(1-c_1^2y^2)}}=\frac{dz}{\sqrt{(1-z^2)(1-c_1^2z^2)}}+\frac{dv}{\sqrt{(1-v^2)(1-c_1^2v^2)}}$$

which is satisfied by $y=\frac{z\sqrt{(1-v^2)(1-c_1^2v^2)}+v\sqrt{(1-z^2)(1-c_1^2z^2)}}{1-c_1^2z^2v^2}$, algebraic function of x. Abel gives a translation of his general theorem in Legendre's notations (which were adopted by Jacobi).

The case where $c_1=c$ corresponds to complex multiplication. Then $a=\frac{m}{\nu}+\frac{m'}{2\nu}\frac{\varpi}{\omega}\sqrt{-1}=\frac{\mu'}{\nu'}-\frac{2\mu}{\nu'}\frac{\omega}{\varpi}\sqrt{-1}$ and $\frac{m}{\nu}=\frac{\mu'}{\nu'}$, $\frac{m'}{2\nu}\frac{\varpi}{\omega}=-\frac{2\mu}{\nu'}\frac{\omega}{\varpi}$. The multiplicator a is real if $m'=\mu=0$ but otherwise we must impose $\frac{\omega}{\varpi}=\frac{1}{2}\sqrt{-\frac{m'\nu'}{\mu\nu}}=\sqrt{k}$ where k is positive rational and we have $a=\delta+\delta'\sqrt{k}\sqrt{-1}$ where δ,δ' are rational numbers. Doing $\alpha=\frac{\omega}{2}$ in (98), Abel obtains

$$\sqrt[4]{c}=\frac{1-e^{-\pi\sqrt{k}}}{1+e^{-\pi\sqrt{k}}}\frac{1-e^{-3\pi\sqrt{k}}}{1+e^{-3\pi\sqrt{k}}}\frac{1-e^{-5\pi\sqrt{k}}}{1+e^{-5\pi\sqrt{k}}}\cdots \quad \text{and}$$

$$\sqrt[4]{b}=\frac{1-e^{-\frac{\pi}{\sqrt{k}}}}{1+e^{-\frac{\pi}{\sqrt{k}}}}\frac{1-e^{-\frac{3\pi}{\sqrt{k}}}}{1+e^{-\frac{3\pi}{\sqrt{k}}}}\frac{1-e^{-\frac{5\pi}{\sqrt{k}}}}{1+e^{-\frac{5\pi}{\sqrt{k}}}}\cdots$$

Abel continued the explanation of the theory of transformation in a memoir published in the third volume of *Crelle's Journal* (1828), *Sur le nombre des transformations différentes qu'on peut faire subir à une fonction elliptique par la substitution d'une fonction rationnelle dont le degré est un nombre premier donné* (*Œuvres*, t. I, p. 456–465). He puts

$$\Delta^2=(1-x^2)(1-c^2x^2),\quad \Delta'^2=(1-y^2)(1-c'^2x^2)$$

and supposes that the differential equation $\frac{dy}{\Delta'}=a\frac{dx}{\Delta}$ is satisfied by

$$y=\frac{A_0+A_1x+\ldots+A_{2n+1}x^{2n+1}}{B_0+B_1x+\ldots+B_{2n+1}x^{2n+1}}$$

where $2n+1$ is a prime number and one of the coefficients A_{2n+1},B_{2n+1} is different from 0. He recalls that, according to the *Solution d'un problème général*, when $B_{2n+1}=0$, one has $y=\frac{\delta}{\varepsilon}\frac{p}{v}$, $c'=\varepsilon^2$ and $a=\frac{\delta}{\varepsilon}$ where $p=x\left(1-\frac{x^2}{\lambda^2\alpha}\right)\cdots\left(1-\frac{x^2}{\lambda^2(n\alpha)}\right)$, $v=(1-c^2\lambda^2\alpha x^2)\cdots(1-c^2\lambda^2(n\alpha)x^2)$, $\varepsilon=c^{n+\frac{1}{2}}\left(\lambda\left(\frac{\omega}{2}+\alpha\right)\ldots\lambda\left(\frac{\omega}{2}+n\alpha\right)\right)^2$, $\delta=c^{n+\frac{1}{2}}(\lambda\alpha\cdot\lambda(2\alpha)\cdots\lambda(n\alpha))^2$ and $\alpha=\frac{m\omega+m'\omega'}{2n+1}$ (m,m' integers). Other solutions are given by composing with $\frac{f'+fy}{g'+gy}$ where f',f,g,g' are constants such that $\left(1+\frac{g+f}{g'+f'}x\right)\left(1+\frac{g-f}{g'-f'}x\right)\left(1+\frac{g+c'f}{g'+c'f'}x\right)\times\left(1+\frac{g-c'f}{g'-c'f'}x\right)=(1-x^2)(1-c'^2x^2)$. Thus, disregarding the signs, one finds 12 values for y and 6 values for c' for each choice of α:

	I	II	III	IV	V	VI
c'	ε^2	$\frac{1}{\varepsilon^2}$	$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2$	$\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2$	$\left(\frac{1-\varepsilon i}{1+\varepsilon i}\right)^2$	$\left(\frac{1+\varepsilon i}{1-\varepsilon i}\right)^2$
a	$\pm\frac{\delta}{\varepsilon}$	$\mp\delta\varepsilon$	$\pm\frac{\delta}{2\varepsilon}(1+\varepsilon)^2 i$	$\mp\frac{\delta}{2\varepsilon}(1-\varepsilon)^2 i$	$\pm\frac{\delta}{2\varepsilon}(1+\varepsilon i)^2 i$	$\mp\frac{\delta}{2\varepsilon}(1-\varepsilon i)^2 i$
y	$\begin{cases}\frac{\delta}{\varepsilon}\frac{p}{v}\\ \frac{1}{\delta\varepsilon}\frac{v}{p}\end{cases}$	$\begin{matrix}\frac{\varepsilon}{\delta}\frac{v}{p}\\ \delta\varepsilon\frac{p}{v}\end{matrix}$	$\frac{1+\varepsilon}{1-\varepsilon}\frac{v\pm\delta p}{v\mp\delta p}$	$\frac{1-\varepsilon}{1+\varepsilon}\frac{v\pm\delta p}{v\mp\delta p}$	$\frac{1+\varepsilon i}{1-\varepsilon i}\frac{v\pm\delta pi}{v\mp\delta pi}$	$\frac{1-\varepsilon i}{1+\varepsilon i}\frac{v\pm\delta pi}{v\mp\delta pi}$.

This comes from the fact that the modulus c^2 is a modular function of level 2 in Klein's sense: it is invariant by the group of 2×2 matrices congruent to the identity *modulo* 2 operating on the ratio of the periods, and this group is of index 6 in SL(2,**Z**).

It remains to count the number of α leading to different solutions. If α and α' lead to the same solution of type I, one finds that $p' = p$, $v' = v$ and $\frac{\delta'}{\varepsilon'} = \pm\frac{\delta}{\varepsilon}$. From $p' = p$ it results that $\lambda^2\alpha' = \lambda^2(\mu\alpha)$ for an integer μ between 1 and n, thus $\alpha' = k\omega + k'\omega' \pm \mu\alpha$ where k, k' are integers. For such a value of α', $p' = p$, $v' = v$, $\delta' = \delta$, and $\varepsilon' = \varepsilon$ and both solutions are effectively equal. Now when $\alpha = \frac{m\omega}{2n+1}$, there exist integers k, μ such that $k(2n+1) \pm \mu m = 1$ and one has $k\omega \pm \mu\alpha = \frac{\omega}{2n+1}$. When $\alpha = \frac{m\omega+m'\omega'}{2n+1}$ with $m' \neq 0$, there exist integers k', μ such that $k'(2n+1) \pm m'\mu = 1$ and one has $k\omega + k'\omega' \pm \mu\alpha = \frac{\omega'+\nu\omega}{2n+1}$ where $\nu = k(2n+1) \pm \mu m$. Thus the different choices for α are $\frac{\omega}{2n+1}, \frac{\omega'}{2n+1}, \frac{\omega'+\omega}{2n+1}, \frac{\omega'+2\omega}{2n+1}, \dots, \frac{\omega'+2n\omega}{2n+1}$; their number is $2n+2$.

The values of y of types III, IV, V and VI may be written in another way with the help of the identities $v - \delta p = (1 - x\sqrt{c})(1 - 2k_1 x\sqrt{c} + cx^2)(1 - 2k_2 x\sqrt{c} + cx^2) \cdots (1 - 2k_n x\sqrt{c} + cx^2)$,

$$v - \delta p\sqrt{-1} = (1 - x\sqrt{-c})(1 - 2k_1' x\sqrt{-c} - cx^2)(1 - 2k_2' x\sqrt{-c} - cx^2) \cdots \times (1 - 2k_n' x\sqrt{-c} - cx^2)$$

and similar expressions for $v + \delta p$, $v + \delta p\sqrt{-1}$, where $k_\mu = \frac{\Delta(\mu\alpha)}{1-c\lambda^2(\mu a)}$, $k_\mu' = \frac{\Delta(\mu\alpha)}{1+c\lambda^2(\mu a)}$, $\Delta(\theta) = \pm\sqrt{(1-\lambda^2\theta)(1-c^2\lambda^2\theta)}$. When $0 < c < 1$, Abel explains that the only transformations for which c' is real correspond to $\alpha = \frac{\omega}{2n+1}$ or $\frac{\omega'-\omega}{2n+1}$ and that they are of type I, II, III or IV.

As we saw above, when $0 < c < 1$, ω is real and $\omega' = \omega + \varpi\sqrt{-1}$ where ϖ is real. Abel gives an expression of $\lambda\theta = f\left(b\left(\frac{\omega}{2} - \theta\right)\right)$ in infinite product

$$\lambda\theta = \frac{2}{\sqrt{c}}\sqrt[4]{q}\sin\left(\frac{\pi}{\omega}\theta\right)\frac{\left(1 - 2q^2\cos\left(\frac{2\pi}{\omega}\theta\right) + q^4\right)\left(1 - 2q^4\cos\left(\frac{2\pi}{\omega}\theta\right) + q^8\right)\cdots}{\left(1 - 2q\cos\left(\frac{2\pi}{\omega}\theta\right) + q^2\right)\left(1 - 2q^3\cos\left(\frac{2\pi}{\omega}\theta\right) + q^6\right)\cdots}$$

where $q = e^{-\frac{\varpi}{\omega}\pi}$ and computes ε with the help of this formula. If $\alpha = \frac{\omega}{2n+1}$, he finds

$$\varepsilon = 2\sqrt[4]{q^{2n+1}}\left(\frac{1+q^{2(2n+1)}}{1+q^{2n+1}}\frac{1+q^{4(2n+1)}}{1+q^{3(2n+1)}}\cdots\right)^2.$$

The other values of α are of the form $\frac{\varpi i+2\mu\omega}{2n+1}$ $(0 \le \mu \le 2n)$ and give

$$\varepsilon = 2\sqrt[4]{\delta_1^{\mu} q^{\frac{1}{2n+1}}} \left(\frac{1+\left(\delta_1^{\mu} q^{\frac{1}{2n+1}}\right)^2}{1+\delta_1^{\mu} q^{\frac{1}{2n+1}}} \frac{1+\left(\delta_1^{\mu} q^{\frac{1}{2n+1}}\right)^4}{1+\left(\delta_1^{\mu} q^{\frac{1}{2n+1}}\right)^3} \cdots \right)^2$$

where $\delta_1 = \cos\frac{2\pi}{2n+1} + \sqrt{-1}\sin\frac{2\pi}{2n+1}$ is a primitive $(2n+1)$-th root of 1. Thus the $2n+2$ values of ε are obtained by replacing q in the expression $2\sqrt[4]{q}\left(\frac{1+q^2}{1+q}\frac{1+q^4}{1+q^3}\cdots\frac{1+q^{2m}}{1+q^{2m-1}}\cdots\right)^2 = \sqrt{c}$ by

$$q^{2n+1}, q^{\frac{1}{2n+1}}, \delta_1 q^{\frac{1}{2n+1}}, \delta_1^2 q^{\frac{1}{2n+1}}, \ldots, \delta_1^{2n} q^{\frac{1}{2n+1}}.$$

The same substitutions in the expression $2\frac{\pi}{\omega}\sqrt[4]{q}\left(\frac{1-q^2}{1-q}\frac{1-q^4}{1-q^3}\cdots\right)^2$ give the $2n+1$ values of δ. Jacobi independently discovered similar rules for the transformations (1829).

In a very short paper published in *Crelle's Journal*(vol. 3, 1828; *Œuvres*, t. I, p. 466), Abel states the rule for the transformation of elliptic integrals of the third kind. Let $f(y,x) = 0$ be an algebraic integral of the differential equation $\frac{dy}{\sqrt{(1-y^2)(1-c'y^2)}} = a\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$. Then $\int \frac{A+Bx^2}{1-\frac{x^2}{n^2}}\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}} = \int \frac{A'+B'y^2}{1-\frac{y^2}{m^2}}\frac{dy}{\sqrt{(1-y^2)(1-c'^2y^2)}} + k\log p$ where A', B', m and k are functions of A, B, n and p is an algebraic function of y and x. The transformed parameter m is determined by the equation $f(m,n) = 0$. For n infinite, the integrals are of the second kind and the rule for the transformation is

$$\int (A+Bx^2)\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}} = \int (A'+B'y^2)\frac{dy}{\sqrt{(1-y^2)(1-c'^2y^2)}} + v$$

where v is an algebraic function of x and y.

8 Further Development of the Theory of Elliptic Functions and Abelian Integrals

In the fourth volume of *Crelle's Journal* (1829; *Œuvres*, t. I, p. 467–477), Abel published a *Note sur quelques formules elliptiques*, devoted to the translation of the formulae given in the the *Recherches* into Legendre's notation in order to recover results published by Jacobi. Supposing that $c = 1$, the problem is to pass from the case in which e is real to the case in which e^2 is negative. Abel puts $\lambda\alpha = f\left(\frac{\omega}{2} - b\alpha\right)$ where $b = \frac{1}{\sqrt{1+e^2}}$ so that $x = \lambda\alpha$ is equivalent to $\alpha = \int_0 \frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$ where $c = \frac{e}{\sqrt{1+e^2}}$ and $b = \sqrt{1-c^2}$. One has

$$\frac{\omega}{2} = b\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}, \quad \frac{\varpi}{2} = b\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-b^2x^2)}}.$$

Abel auxiliary functions $\lambda'\alpha = \sqrt{1-\lambda^2\alpha}$, $\lambda''\alpha = \sqrt{1-c^2\lambda^2\alpha}$ to play the roles of cosamα, Δamα in Jacobi's notation. One has $\lambda'\alpha = \varphi\left(\frac{\omega}{2} - b\alpha\right)$, $\lambda''\alpha = bF\left(\frac{\omega}{2} - b\alpha\right)$. As in the preceding papers, Abel uses the expansions of $f\alpha$, $\varphi\alpha$ and $F\alpha$ in simple infinite products to obtain expressions for $\lambda\theta$, $\lambda'\theta$ and $\lambda''\theta$ (cf. (98)):

$$\begin{aligned}
\lambda\theta &= A\frac{1-\rho^2}{1+\rho^2}\frac{1-\rho^2r^2}{1+\rho^2r^2}\frac{1-\rho^{-2}r^2}{1+\rho^{-2}r^2}\frac{1-\rho^2r^4}{1+\rho^2r^4}\frac{1-\rho^{-2}r^4}{1+\rho^{-2}r^4}\cdots,\\
\lambda'\theta &= A'\frac{2\rho}{1+\rho^2}\frac{1-\rho^2r}{1+\rho^2r^2}\frac{1-\rho^{-2}r}{1+\rho^{-2}r^2}\frac{1-\rho^2r^3}{1+\rho^2r^4}\frac{1-\rho^{-2}r^3}{1+\rho^{-2}r^4}\cdots,\\
\lambda''\theta &= A''\frac{2\rho}{1+\rho^2}\frac{1+\rho^2r}{1+\rho^2r^2}\frac{1+\rho^{-2}r}{1+\rho^{-2}r^2}\frac{1+\rho^2r^3}{1+\rho^2r^4}.\frac{1+\rho^{-2}r^3}{1+\rho^{-2}r^4}\cdots
\end{aligned} \tag{100}$$

where $\rho = e^{-\frac{\theta\pi}{\varpi'}}$, $r = e^{-\frac{\omega'}{\varpi'}\pi}$, $\frac{\omega'}{2} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-c^2\sin^2\theta}}$, $\frac{\varpi'}{2} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-b^2\sin^2\theta}}$ and

$$\begin{aligned}
\sqrt{A} &= \frac{(1+r)(1+r^3)\cdots}{(1-r)(1-r^3)\cdots},\\
\sqrt{A'} &= \frac{(1+r^2)(1+r^4)(1+r^6)\cdots}{(1-r)(1-r^3)(1-r^5)\cdots},\\
\sqrt{A''} &= \frac{(1+r^2)(1+r^4)(1+r^6)\cdots}{(1+r)(1+r^3)(1+r^5)\cdots}.
\end{aligned} \tag{101}$$

Doing $\theta = \frac{\omega'}{2} + \frac{\varpi'}{2}i$ in (100), one obtains $\rho^2 = -r$ and

$$\begin{aligned}
\lambda\theta &= f\left(\frac{\varpi}{2}i\right) = \frac{\sqrt{1+e^2}}{e} = \frac{1}{c} = A\left(\frac{(1+r)(1+r^3)(1+r^5)\dots}{(1-r)(1-r^3)(1-r^5)\dots}\right)^2 = A^2,\\
\lambda'\theta &= -\varphi\left(\frac{\varpi i}{2}\right) = -\frac{i}{e} = i\frac{b}{c} = 4A'i\sqrt{r}\left(\frac{1+r^2}{1-r}\frac{1+r^4}{1-r^3}\cdots\right)^2.
\end{aligned}$$

Thus $A = \frac{1}{\sqrt{c}}$, $A' = \frac{1}{2\sqrt[4]{r}}\sqrt{\frac{b}{c}}$. In a similar way, doing $\theta = \frac{\omega'}{2}$ one obtains $\rho^2 = r$ and $\lambda''\theta = b = 4A''\sqrt{r}\left(\frac{1+r^2}{1+r}\frac{1+r^4}{1+r^3}\dots\right)^2 = 4A''\sqrt{r}\cdot A''$ so that $A'' = \frac{\sqrt{b}}{2\sqrt[4]{r}}$. These values compared with (101) give

$$\begin{aligned}
\sqrt[4]{c} &= \frac{1-r}{1+r}\frac{1-r^3}{1+r^3}\frac{1-r^5}{1+r^5}\cdots,\\
\sqrt[4]{\frac{b}{c}} &= \sqrt{2}\sqrt[8]{r}\frac{1+r^2}{1-r}\frac{1+r^4}{1-r^3}\frac{1+r^6}{1-r^5}\cdots,\\
\sqrt[4]{b} &= \sqrt{2}\sqrt[8]{r}\frac{1+r^2}{1+r}\frac{1+r^4}{1+r^3}\frac{1+r^6}{1+r^5}\cdots.
\end{aligned} \tag{102}$$

The limit value of $\frac{\lambda\theta}{1-\rho^2}$ for $\theta \to 0$ is $\frac{\varpi'}{2\pi}$ and, comparing with (100), this gives

$$\sqrt[4]{c}\sqrt{\frac{\varpi'}{\pi}} = \frac{(1-r^2)(1-r^4)(1-r^6)\cdots}{(1+r^2)(1+r^4)(1+r^6)\cdots} \text{ and}$$
$$\sqrt{\frac{\varpi'}{\pi}} = \frac{(1+r)(1-r^2)(1+r^3)(1-r^4)\cdots}{(1-r)(1+r^2)(1-r^3)(1+r^4)\cdots}$$
$$= ((1+r)(1+r^3)(1+r^5)\cdots)^2$$
$$\times(1+r)(1+r^2)(1+r^3)\cdots\times(1-r)(1-r^2)(1-r^3)\cdots.$$

Abel puts $P = (1+r)(1+r^3)(1+r^5)\cdots$ and $P' = (1+r^2)(1+r^4)(1+r^6)\cdots$ so that

$$PP' = (1+r)(1+r^2)(1+r^3)\cdots = \frac{1}{(1-r)(1-r^3)(1-r^5)\cdots}$$

and $\sqrt[4]{c} = \frac{1}{P^2P'}$, $\sqrt[4]{b} = \sqrt{2}\sqrt[8]{r}\frac{P'}{P}$. From these relations, he draws

$$P = \sqrt[6]{2}\sqrt[24]{\frac{r}{b^2c^2}}, \quad P' = \frac{\sqrt[6]{b}\sqrt[24]{r}}{\sqrt[3]{2}\sqrt[12]{c}}\frac{1}{\sqrt[8]{r}}, \tag{103}$$

$PP' = (1+r)(1+r^2)(1+r^3)(1+r^4)\cdots = \frac{\sqrt[12]{b}}{\sqrt[6]{2c}\sqrt[24]{r}}$, $(1-r)(1-r^2)(1-r^3)\cdots = \frac{\sqrt[12]{b}\sqrt[3]{c}}{\sqrt[6]{2}\sqrt[24]{r}}\sqrt{\frac{\varpi'}{\pi}}$, one of the formulae published by Jacobi.

Now putting $q = e^{-\frac{\varpi'}{\omega'}\pi}$ so that $\log r \log q = \pi^2$, $\theta = \frac{\varpi'}{2} + \frac{\omega'}{2}\sqrt{-1} + \frac{\omega'}{\pi}x\sqrt{-1}$ and exchanging b and c, Abel obtains from (100)

$$\lambda\left(\frac{\omega'}{\pi}x\right) = \frac{2}{\sqrt{c}}\sqrt[4]{q}\sin x\frac{1-2q^2\cos 2x+q^4}{1-2q\cos 2x+q^2}\frac{1-2q^4\cos 2x+q^8}{1-2q^3\cos 2x+q^6}\cdots,$$
$$\lambda'\left(\frac{\omega'}{\pi}x\right) = 2\sqrt{\frac{b}{c}}\sqrt[4]{q}\cos x\frac{1+2q^2\cos 2x+q^4}{1-2q\cos 2x+q^2}\frac{1+2q^4\cos 2x+q^8}{1-2q^3\cos 2x+q^6}\cdots,$$
$$\lambda''\left(\frac{\omega'}{\pi}x\right) = \sqrt{b}\frac{1+2q\cos 2x+q^2}{1-2q\cos 2x+q^2}\frac{1+2q^3\cos 2x+q^6}{1-2q^3\cos 2x+q^6}\cdots. \tag{104}$$

By comparison with Jacobi's formula for $\Delta\mathrm{am}\alpha$, Abel finds

$$\frac{1+2q\cos 2x+2q^4\cos 4x+2q^9\cos 6x+\ldots}{1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+\ldots}$$
$$= \frac{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\ldots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\ldots}.$$

The logarithms of (104) are written

$$\log\lambda\left(\frac{\omega'}{\pi}x\right) = \log 2 - \frac{1}{2}\log c - \frac{1}{4}\frac{\varpi'}{\omega'}\pi + \log\sin x$$
$$+2\left(\cos 2x\frac{q}{1+q} + \frac{1}{2}\cos 4x\frac{q^2}{1+q^2} + \frac{1}{3}\cos 6x\frac{q^3}{1+q^3} + \dots\right),$$
$$\log\lambda'\left(\frac{\omega'}{\pi}x\right) = \log 2 + \frac{1}{2}\log b - \frac{1}{2}\log c - \frac{1}{4}\frac{\varpi'}{\omega'}\pi + \log\cos x$$
$$+2\left(\cos 2x\frac{q}{1-q} + \frac{1}{2}\cos 4x\frac{q^2}{1+q^2} + \frac{1}{3}\cos 6x\frac{q^3}{1-q^3} + \dots\right),$$
$$\log\lambda''\left(\frac{\omega'}{\pi}x\right) = \frac{1}{2}\log b + 4\left(\cos 2x\frac{q}{1-q^2} + \frac{1}{3}\cos 6x\frac{q^3}{1-q^6} + \dots\right).$$

For $x = 0$, this last formula gives $\log\left(\frac{1}{b}\right) = 8\left(\frac{q}{1-q^2} + \frac{1}{3}\frac{q^3}{1-q^6} + \frac{1}{5}\frac{q^5}{1-q^{10}} + \dots\right)$ and the first one gives

$$\log\left(\frac{1}{c}\right) = \frac{1}{2}\frac{\varpi'}{\omega'}\pi - 2\log 2 + 4\left(\frac{q}{1+q} - \frac{1}{2}\frac{q^2}{1+q^2} + \frac{1}{3}\frac{q^3}{1+q^3} - \dots\right)$$

which is also equal to $8\left(\frac{r}{1-r^2} + \frac{1}{3}\frac{r^3}{1-r^6} + \frac{1}{5}\frac{r^5}{1-r^{10}} + \dots\right)$ according to (102). From the expansions of $\varphi\left(\alpha\frac{\omega}{2}\right)$, $f\left(\alpha\frac{\omega}{2}\right)$ and $F\left(\alpha\frac{\omega}{2}\right)$ in simple series (*Recherches*, formulae (86)), Abel deduces

$$\lambda\left(\frac{\omega'}{\pi}x\right) = \frac{4\pi}{c\omega'}\sqrt{q}\left(\sin x\frac{1}{1-q} + \sin 3x\frac{q}{1-q^3} + \sin 5x\frac{q^2}{1-q^5} + \dots\right)$$
$$\lambda'\left(\frac{\omega'}{\pi}x\right) = \frac{4\pi}{c\omega'}\sqrt{q}\left(\cos x\frac{1}{1+q} + \cos 3x\frac{q}{1+q^3} + \cos 5x\frac{q^2}{1+q^5} + \dots\right)$$
$$= \frac{2\pi}{c\varpi'}\left(\frac{r^x - r^{1-x}}{1+r} - \frac{r^{3x} - r^{3-3x}}{1+r^3} + \frac{r^{5x} - r^{5-5x}}{1+r^5} + \dots\right),$$
$$\lambda''\left(\frac{\omega'}{\pi}x\right) = \frac{2\pi}{\varpi'}\left(\frac{r^x + r^{1-x}}{1-r} - \frac{r^{3x} + r^{3-3x}}{1-r^3} + \frac{r^{5x} + r^{5-5x}}{1-r^5} + \dots\right).$$

Let c' be a modulus (between 0 and 1) such that there exists a transformation from the elliptic functions of modulus c to those of modulus c', and let ω'', ϖ'', r', q' be associated to c' as ω', ϖ', r, q are associated to c. The characterisation stated in *Solution d'un problème général* is $\frac{\omega''}{\varpi''} = \frac{n}{m}\frac{\omega'}{\varpi'}$ where n, m are integers, or $r' = r^{\frac{n}{m}}$, $q' = q^{\frac{m}{n}}$. For instance, let us take $c = \sqrt{\frac{1}{2}}$, so that $\varpi' = \omega'$ and $r = e^{-\pi}$. Any admissible value of c' is given by

$$\sqrt[4]{c'} = \frac{1-e^{-\mu\pi}}{1+e^{-\mu\pi}}\frac{1-e^{-3\mu\pi}}{1+e^{-3\mu\pi}}\frac{1-e^{-5\mu\pi}}{1+e^{-5\mu\pi}}\cdots$$
$$= \sqrt{2}e^{-\frac{\pi}{8\mu}}\frac{1+e^{-\frac{2\pi}{\mu}}}{1+e^{-\frac{\pi}{\mu}}}\frac{1+e^{-\frac{4\pi}{\mu}}}{1+e^{-\frac{3\pi}{\mu}}}\frac{1+e^{-\frac{6\pi}{\mu}}}{1+e^{-\frac{5\pi}{\mu}}}\cdots$$

where μ is a rational number and such a c' is expressible by radicals. Another example is that in which $b' = c$ or $c' = b$; then $\omega'' = \varpi'$, $\varpi'' = \omega'$. In this case $\frac{\varpi'}{\omega'} = \frac{\omega''}{\varpi''} = \frac{n}{m}\frac{\omega'}{\varpi'}$ and $\frac{\omega'}{\varpi'} = \sqrt{\frac{m}{n}} = \sqrt{\mu}$. Thus $\sqrt[4]{c} = \frac{1-e^{-\pi\sqrt{\mu}}}{1+e^{-\pi\sqrt{\mu}}}\frac{1-e^{-3\pi\sqrt{\mu}}}{1+e^{-3\pi\sqrt{\mu}}}\frac{1-e^{-5\pi\sqrt{\mu}}}{1+e^{-5\pi\sqrt{\mu}}}\cdots$ and $\sqrt[4]{b} = \frac{1-e^{-\frac{\pi}{\sqrt{\mu}}}}{1+e^{-\frac{\pi}{\sqrt{\mu}}}}\frac{1-e^{-\frac{3\pi}{\sqrt{\mu}}}}{1+e^{-\frac{3\pi}{\sqrt{\mu}}}}\frac{1-e^{-\frac{5\pi}{\sqrt{\mu}}}}{1+e^{-\frac{5\pi}{\sqrt{\mu}}}}\cdots$

At the end of this paper, Abel deduces the functional equation for a *theta*-function from (103). Exchanging c and b and r and q, he obtains $(1+q)(1+q^3)(1+q^5)\cdots = \sqrt[6]{2}\frac{\sqrt[24]{q}}{\sqrt[12]{bc}}$ and comparing with (103)

$$\frac{1}{\sqrt[24]{r}}(1+r)(1+r^3)(1+r^5)\cdots = \frac{1}{\sqrt[24]{q}}(1+q)(1+q^3)(1+q^5)\cdots$$

whenever r and q are between 0 and 1 and related by $\log r \cdot \log q = \pi^2$. He recalls some other results due to Cauchy (1818) and to Jacobi (1829).

In a second paper of the fourth volume of *Crelle's Journal* (1829), *Théorèmes sur les fonctions elliptiques* (*Œuvres*, t. I, p. 508–514), Abel considers the equation $\varphi(2n+1)\theta = R$ of which the roots are $x = \varphi(\theta + m\alpha + \mu\beta)$ where φ is the elliptic function of the *Recherches*, $\alpha = \frac{2\omega}{2n+1}$, $\beta = \frac{2\varpi i}{2n+1}$ and m, μ are integers. He proves that if $\psi\theta$ is a polynomial in these roots which is invariant when θ is changed into $\theta+\alpha$ or into $\theta+\beta$, one has

$$\psi\theta = p + q f(2n+1)\theta \cdot F(2n+1)\theta$$

where p and q are polynomials in $\varphi(2n+1)\theta$, of respective degrees ν and $\nu-2$, ν being the highest exponent of $\varphi\theta$ in $\psi\theta$. Indeed, by the addition theorem (73), $\varphi(\theta + m\alpha + \mu\beta)$ is a rational function of $\varphi\theta$ and $f\theta \cdot F\theta$. Since $(f\theta \cdot F\theta)^2 = (1 - c^2\varphi^2\theta)(1 + e^2\varphi^2\theta)$, one has

$$\psi\theta = \psi_1(\varphi\theta) + \psi_2(\varphi\theta) f\theta \cdot F\theta$$

where $\psi_1(\varphi\theta)$ and $\psi_2(\varphi\theta)$ are rational. They are respectively given by

$$\psi_1(\varphi\theta) = \frac{1}{2}(\psi\theta + \psi(\omega-\theta)) \quad \text{and}$$
$$\psi_2(\varphi\theta) f\theta \cdot F\theta = \frac{1}{2}(\psi\theta - \psi(\omega-\theta)). \tag{105}$$

The invariance of $\psi\theta$ by $\theta \mapsto \theta+\alpha$ or $\theta+\beta$ implies that $\psi_1(\varphi(\theta + m\alpha + \mu\beta)) = \psi_1(\varphi\theta)$, so that $\psi_1(\varphi\theta)$ is a rational symmetric function of the roots of the considered equation. Thus $\psi_1(\varphi\theta) = p$ rational function of $\varphi(2n+1)\theta = y$. If $y = \varphi(2n+1)\delta$ is a pole of p, (105) shows that some $\delta + m\alpha + \mu\beta$ or some $\omega - \delta + m\alpha + \mu\beta$ is a pole of φ, but then $(2n+1)\delta$ is also a pole of φ, which is absurd. On the other hand, $f(2n+1)\theta = f\theta \cdot u$, $F(2n+1)\theta = F\theta \cdot v$ where u and v are rational functions of $\varphi\theta$. It results that $\frac{\psi_2(\varphi\theta) f\theta \cdot F\theta}{f(2n+1)\theta \cdot F(2n+1)\theta} = \chi(\varphi\theta)$ rational function of $\varphi\theta$ also equal to $\frac{1}{2}\frac{\psi\theta - \psi(\omega-\theta)}{f(2n+1)\theta \cdot F(2n+1)\theta}$ according to (105). Thus $\chi(\varphi\theta)$ is invariant by $\theta \mapsto \theta+\alpha$ or $\theta+\beta$

and one proves as above that is it a polynomial q in $\varphi(2n+1)\theta$. Abel computes the degrees of p and q by considering the behaviour of $\psi\theta$ and $\psi(\omega-\theta)$ when $\varphi\theta$ is infinite.

When $\nu = 1$, p is of degree 1 and $q = 0$, so that $\psi\theta = A + B\varphi(2n+1)\theta$ where A and B are constants. This is the case for $\psi\theta = \sum_{m=0}^{2n}\sum_{\mu=0}^{2n}\pi(\theta + m\alpha + \mu\beta)$ where $\pi\theta$ is the product of some roots of the equation and one finds that $A = 0$ when the number of factors of $\pi\theta$ is odd whereas $B = 0$ when this number is even.

In the same way, Abel obtains that if $\psi\theta$ is a polynomial in the quantities $f(\theta + m\alpha + \mu\beta)$ (resp. $F(\theta + m\alpha + \mu\beta)$) such that $\psi(\theta) = \psi(\theta+\alpha) = \psi(\theta+\beta)$, then $\psi\theta = p + q\varphi(2n+1)\theta \cdot F(2n+1)\theta$ (resp. $p + q\varphi(2n+1)\theta \cdot f(2n+1)\theta$) where p and q are polynomials in $f(2n+1)\theta$ (resp. $F(2n+1)\theta$) of respective degrees ν, $\nu - 2$, ν beeing the highest exponent of $f\theta$ (resp. $F\theta$) in $\psi\theta$.

As an application, Abel deduces a formula established by Jacobi (1828) for the division of elliptic integrals: $\varphi\left(\frac{\theta}{2n+1}\right) = \frac{1}{2n+1}\sum_{m=0}^{4n^2+4n}\sqrt[2n+1]{p_m + q_m f\theta \cdot F\theta}$ where p_m (resp. q_m) is an odd (resp. even) polynomial in $\varphi\theta$ of degree $2n+1$ (resp. $2n-2$) and $p_m^2 - q_m^2(f\theta)^2(F\theta)^2 = (\varphi^2\theta - a_m^2)^{2n+1}$ where a_m is a constant.

A third memoir of Abel in the volume 4 of *Crelle's Journal* (1829) is a small treatise on elliptic functions, titled *Précis d'une théorie des fonctions elliptiques* (*Œuvres*, t. I, p. 518–617). He uses the following notations: $\Delta(x,c) = \pm\sqrt{(1-x^2)(1-c^2x^2)}$, $\varpi(x,c) = \int \frac{dx}{\Delta(x,c)}$ (integral of the first kind), $\varpi_0(x,c) = \int \frac{x^2dx}{\Delta(x,c)}$ (integral of the second kind) and

$$\Pi(x,c,a) = \int \frac{dx}{\left(1-\frac{x^2}{a^2}\right)\Delta(x,c)} \quad \text{(integral of the third kind).}$$

The general problem dealt with by Abel is the following: "To find all the possible cases in which one may satisfy an equation of the form

$$\begin{aligned} &\alpha_1\varpi(x_1,c_1) + \alpha_2\varpi(x_2,c_2) + \ldots + \alpha_n\varpi(x_n,c_n) \\ &+\alpha'_1\varpi_0(x'_1,c'_1) + \alpha'_2\varpi_0(x'_2,c'_2) + \ldots + \alpha'_m\varpi_0(x'_m,c'_m) \\ &+\alpha''_1\Pi(x''_1,c''_1,a_1) + \alpha''_2\Pi(x''_2,c''_2,a_2) + \ldots + \alpha''_\mu\Pi(x''_\mu,c''_\mu,a_\mu) \\ &= u + A_1\log v_1 + A_2\log v_2 + \ldots + A_\nu\log v_\nu \end{aligned} \tag{106}$$

where $\alpha_1,\alpha_2,\ldots,\alpha_n$; $\alpha'_1,\alpha'_2,\ldots,\alpha'_m$; $\alpha''_1,\alpha''_2,\ldots,\alpha''_\mu$; $A_1,A_2,\ldots,A_\nu$ are constant quantities, $x_1,x_2,\ldots,x_n$; $x'_1,x'_2,\ldots,x'_m$; $x''_1,x''_2,\ldots,x''_\mu$ variables related by *algebraic* equations and $u, v_1, v_2, \ldots, v_\nu$ *algebraic* functions of these variables." This problem is attacked by purely algebraic means, that is without the use of elliptic functions and their double periodicity.

Here are some results announced in the introduction: " If $\int \frac{rdx}{\Delta(x,c)}$, where r is an arbitrary rational function of x, is expressible by algebraic and logarithmic functions and by elliptic integrals $\psi, \psi_1, \psi_2, \ldots$, one may always suppose that

$$\int \frac{rdx}{\Delta(x,c)} = p\Delta(x,c) + \alpha\psi(y) + \alpha'\psi_1(y_1) + \alpha''\psi_2(y_2) + \dots$$
$$+A_1 \log \frac{q_1 + q_1'\Delta(x,c)}{q_1 - q_1'\Delta(x,c)} + A_2 \log \frac{q_2 + q_2'\Delta(x,c)}{q_2 - q_2'\Delta(x,c)} + \dots$$

where all the quantities $p, q_1, q_2, \dots, q_1', q_2', \dots, y, y_1, y_2, \dots$ are *rational functions* of x." In this statement, $\Delta(x,c)$ may be the square root of a polynomial of any degree.

"If any equation of the form (106) takes place and one designates by c any one of the moduli which figure in it, among the other moduli there is at least one c' such that the differential equation $\frac{dy}{\Delta(y,c')} = \varepsilon\frac{dx}{\Delta(x,c)}$ may be satisfied by putting for y a *rational* function of x, and *vice versa*."

The second part of the memoir was not written by Abel and we have only the statement of its principal results in the introduction. Abel supposes that $0 < c < 1$ and introduces the elliptic function $\lambda\theta$ inverse of $\varpi(x,c)$, with its main properties: double periodicity, with the fundamental periods 2ϖ, ωi given by $\frac{\varpi}{2} = \int_0^1 \frac{dx}{\Delta(x,c)}$, $\frac{\omega}{2} = \int_0^1 \frac{dx}{\Delta(x,b)}$, determination of its zeros and poles, equation $\lambda(\theta' + \theta)\lambda(\theta' - \theta) = \frac{\lambda^2\theta' - \lambda^2\theta}{1 - c^2\lambda^2\theta\cdot\lambda^2\theta'}$, expansion in infinite product. He proves that if the equation $(\lambda\theta)^{2n} + a_{n-1}(\lambda\theta)^{2n-2} + \dots + a_1(\lambda\theta)^2 + a_0 = (b_0\lambda\theta + b_1(\lambda\theta)^3 + \dots + b_{n-2}(\lambda\theta)^{2n-3})\Delta(\lambda\theta, c)$ is satisfied by $\theta = \theta_1, \theta_2, \dots, \theta_{2n}$ such that $\lambda^2\theta_1, \lambda^2\theta_2, \dots, \lambda^2\theta_{2n}$ be different between them, then $\lambda(\theta_1 + \theta_2 + \dots + \theta_{2n}) = 0$ and $-\lambda(\theta_{2n}) = \lambda(\theta_1 + \theta_2 + \dots + \theta_{2n-1}) = \frac{-a_0}{\lambda\theta_1\lambda\theta_2\dots\lambda\theta_{2n-1}}$. This statement gives a general theorem for the addition and its proof is given in the first part.

The roots of the equation of division of the periods are related by remarkable linear relations, where $\delta = \cos\frac{2\pi}{2\mu+1} + \sqrt{-1}\sin\frac{2\pi}{2\mu+1}$ is a primitive $(2\mu+1)$-th root of 1:

$$0 = \lambda\left(\frac{2m\varpi}{2\mu+1}\right) + \delta^k\lambda\left(\frac{2m\varpi + \omega i}{2\mu+1}\right) + \delta^{2k}\lambda\left(\frac{2m\varpi + 2\omega i}{2\mu+1}\right)$$
$$+\delta^{3k}\lambda\left(\frac{2m\varpi + 3\omega i}{2\mu+1}\right) + \dots + \delta^{2\mu k}\lambda\left(\frac{2m\varpi + 2\mu\omega i}{2\mu+1}\right),$$
$$0 = \lambda\left(\frac{m\omega i}{2\mu+1}\right) + \delta^{k'}\lambda\left(\frac{2\varpi + m\omega i}{2\mu+1}\right) + \delta^{2k'}\lambda\left(\frac{4\varpi + m\omega i}{2\mu+1}\right)$$
$$+\delta^{3k'}\lambda\left(\frac{6\varpi + m\omega i}{2\mu+1}\right) + \dots + \delta^{2\mu k'}\lambda\left(\frac{4\mu\varpi + m\omega i}{2\mu+1}\right).$$

Sylow gave a demonstration of these relations in 1864 and he explains how to deduce them from the theory of transformation in the final notes to Abel's Works. He also reproduces another proof communicated to him by Kronecker in a letter in 1876 (*Œuvres*, t. II, p. 314–316).

If there is a transformation of $\int \frac{dx}{\Delta(x,c)}$ (with $0 < c < 1$) into $\varepsilon \int \frac{dy}{\Delta(y,c')}$ (with c' arbitrary) by putting for y an algebraic function of x, then c' is given by one of the following formulae: $\sqrt[4]{c'} = \sqrt{2}\sqrt[8]{q_1}\frac{(1+q_1^2)(1+q_1^4)(1+q_1^6)\cdots}{(1+q_1)(1+q_1^3)(1+q_1^5)\cdots}$, $\sqrt[4]{c'} = \frac{1-q_1}{1+q_1}\frac{1-q_1^3}{1+q_1^3}\frac{1-q_1^5}{1+q_1^5}\cdots$ where $q_1 = q^{\mu} = e^{(-\mu\frac{\omega}{\varpi}+\mu' i)\pi}$, μ, μ' rational numbers.

As in the *Solution d'un problème général*, Abel obtains a statement concerning the rational transformation of a real elliptic integral of modulus c into another of modulus c', with $0 < c, c' < 1$. The periods $\varpi, \omega, \varpi', \omega'$ must be related by $\frac{\varpi'}{\omega'} = \frac{n'}{m}\frac{\varpi}{\omega}$ where n', m are integers and this condition is sufficient, the multiplicator being $\varepsilon = m\frac{\varpi'}{\varpi}$. Abel proposes to determine the rational function of x expressing y by means of its zeros and poles.

When c may be transformed into its complement $b = \sqrt{1-c^2}$ (singular modulus), $\frac{\varpi}{\omega} = \sqrt{\frac{m}{n}}$ and $\frac{dy}{\Delta(y,b)} = \sqrt{mn}\frac{dx}{\Delta(x,c)}$. Abel says that c is determined by an algebraic equation which "seems to be solvable by radicals"; he is thus doubtful about this fact, later proved by Kronecker. In the final notes (*Œuvres*, t. II, p. 316–318), Sylow gives a proof of this fact by reduction to the solvability of the equation of division of the periods. Abel gives an expression of $\sqrt[4]{c}$ by an infinite product. He also state that two moduli c and c' which may be transformed into one another are related by an algebraic relation and that, in general, it does not seem possible to draw the value of c' by radicals. But it is possible when c may be transformed into its complement. According to Abel, all the roots of a modular equation are *rationally* expressible by two of them, but this statement is mistaken; they are expressible with the help of radicals by one of them.

Abel gives an expression of $\lambda\theta$ as a quotient of two entire functions $\varphi\theta = \theta + a\theta^3 + a'\theta^5 + \ldots$ and $f\theta = 1 + b'\theta^4 + b''\theta^6 + \ldots$ related by the functional equations

$$\varphi(\theta'+\theta)\varphi(\theta'-\theta) = (\varphi\theta f\theta')^2 - (\varphi\theta f\theta')^2 - (\varphi\theta' f\theta)^2,$$
$$f(\theta'+\theta)f(\theta'-\theta) = (f\theta f\theta')^2 - c^2(\varphi\theta\varphi\theta')^2.$$

These functions are similar to the al-functions of Weierstrass, later replaced by σ. As we have said, Abel communicated the functional equations to Legendre, saying that they characterise the functions φ and f (see §1).

Abel adds that most of these properties are still valid when the modulus c is a complex number.

The first part of the memoir, the only one written and published, is divided in five chapters. In the first one (p. 528–545), Abel deals with the general properties of elliptic integrals, beginning by Euler addition theorem proved as a particular case of Abel theorem: "Let fx and φx be two arbitrary polynomials in x, one even and the other odd, with indeterminate coefficients. Let us put $(fx)^2 - (\varphi x)^2(\Delta x)^2 = A(x^2 - x_1^2)(x^2 - x_2^2)(x^2 - x_3^2)\ldots(x^2 - x_{\mu}^2)$ where A does not depend on x, I say that one will have

$$\Pi x_1 + \Pi x_2 + \Pi x_3 + \ldots + \Pi x_{\mu} = C - \frac{a}{2\Delta a}\log\frac{fa + \varphi a \cdot \Delta a}{fa - \varphi a \cdot \Delta a}, \qquad (107)$$

a denoting the parameter of the function Πx, such that $\Pi x = \int \frac{dx}{\left(1-\frac{x^2}{a^2}\right)\Delta x}$. The quantity C is the integration constant."

When all the coefficients of fx and φx are independent variables, $x_1, x_2, \dots, x_\mu$ are all distinct and one has $\psi x = (fx)^2 - (\varphi x)^2(\Delta x)^2 = 0$ and $fx + \varphi x \Delta x = 0$ if x is any one of them. Let δ denote the differentiation with respect to the variable coefficients of fx and φx. One has

$$\psi' x \cdot dx + 2 fx \cdot \delta fx - 2\varphi x \cdot \delta\varphi x \cdot (\Delta x)^2 = \psi' x \cdot dx - 2\Delta x(\varphi x \cdot \delta fx - fx \cdot \delta\varphi x) = 0,$$

thus $\Pi x = \int \frac{2(\varphi x \cdot \delta fx - fx \cdot \delta \varphi x)}{\left(1 - \frac{x^2}{a^2}\right)\psi' x}$ and

$$\Pi x_1 + \Pi x_2 + \Pi x_3 + \dots + \Pi x_\mu = \int \left(\frac{\theta x_1}{\left(1 - \frac{x_1^2}{a^2}\right)\psi' x_1} + \frac{\theta x_2}{\left(1 - \frac{x_2^2}{a^2}\right)\psi' x_2} + \dots + \frac{\theta x_\mu}{\left(1 - \frac{x_\mu^2}{a^2}\right)\psi' x_\mu} \right) \quad (108)$$

where $\theta x = 2(\varphi x \cdot \delta fx - fx \cdot \delta\varphi x)$ is a polynomial in x, of degree less than that of ψx. Therefore, the right hand side of (108) is equal to $\int \frac{a\theta a}{2\psi a} = a \int \frac{\varphi a \cdot \delta fa - fa \cdot \delta\varphi a}{(fa)^2 - (\varphi a)^2(\Delta a)^2} = C - \frac{a}{2\Delta a} \log \frac{fa + \varphi a \cdot \Delta a}{fa - \varphi a \cdot \Delta a}$. This proof is valid whenever Δx is the square root of an even polynomial in x, as was seen in the publication of Abel theorem for hyperelliptic functions in the third volume of *Crelle's Journal* (see our §5). It is naturally extended to the case in which the coefficients of fx and φx are no more independent and some of the x_j may be equal. Taking a infinite, Πx is reduced to the integral of the first kind ϖx and the logarithmic part vanishes, so that

$$\varpi x_1 + \varpi x_2 + \dots + \varpi x_\mu = C.$$

An expansion of both members of (107) in ascending powers of $\frac{1}{a}$ gives, by comparison of the coefficients of $\frac{1}{a^2}$, $\varpi_0 x_1 + \varpi_0 x_2 + \dots + \varpi_0 x_\mu = C - p$ where p is an *algebraic* function of the variables.

As in the general case of Abel theorem, one may choose $x_1, x_2, \dots, x_m$ as independent variables and determine the coefficients of fx and φx in function of them. The $\mu - m$ quantities $x_{m+1}^2, x_{m+2}^2, \dots, x_\mu^2$ are then the roots of an equation of degree $\mu - m$ and they are algebraic functions of $x_1, x_2, \dots, x_m$. The minimum value of $\mu - m$ is 1. When $\mu = 2n$ is even, one may take

$$\begin{aligned} &fx = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_{n-1} x^{2n-2} + x^{2n}, \\ &\varphi x = (b_0 + b_1 x^2 + b_2 x^4 + \dots + b_{n-2} x^{2n-4})x, \\ &(fx)^2 - (\varphi x)^2(1 - x^2)(1 - c^2 x^2) \\ &\quad = (x^2 - x_1^2)(x^2 - x_2^2)\dots(x^2 - x_{2n-1}^2)(x^2 - y^2), \qquad (109) \\ &fx_1 + \varphi x_1 \cdot \Delta x_1 = fx_2 + \varphi x_2 \cdot \Delta x_2 = \dots \\ &\quad = fx_{2n-1} + \varphi x_{2n-1} \cdot \Delta x_{2n-1} = 0. \qquad (110) \end{aligned}$$

The linear equations of the last line determine the coefficients $a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-2}$ as *rational* functions of $x_1, x_2, \dots, x_{2n-1}, \Delta x_1, \Delta x_2, \dots, \Delta x_{2n-1}$. For $x = 0$, (109) gives

$$a_0^2 = x_1^2 x_2^2 \dots x_{2n-1}^2 y^2 \quad \text{whence } y = -\frac{a_0}{x_1 x_2 \dots x_{2n-1}}.$$

Since $fx_{2n} + \varphi x_{2n} \cdot \Delta x_{2n} = 0$, putting $\Delta x_{2n} = -\Delta y$ one has $\Delta y = \frac{fy}{\varphi y}$ a *rational* function of $x_1, x_2, \dots, \Delta x_1, \Delta x_2, \dots$ as is y. If, in (109), we put $x_1 = x_2 = \dots = x_{2n-1} = 0$, the right hand side becomes divisible by x^{4n-2} and we must have $a_0 = a_1 = \dots = a_{n-1} = b_0 = b_1 = \dots = b_{n-2} = 0$. Thus we obtain $x^{4n} = x^{4n-2}(x^2 - y^2)$ and $y = 0$. Abel shows that if $\Delta x_1 = \Delta x_2 = \dots = \Delta x_{2n-1} = 1$ for $x_1 = x_2 = \dots = x_{2n-1} = 0$, then $\Delta y = 1$. Indeed, for $x_1, x_2, \dots, x_{2n-1}$ infinitely small, the equations (110) reduce to

$$x^{2n} + a_{n-1}x^{2n-2} + b_{n-2}x^{2n-3} + \dots + b_0 x + a_0 = 0, \tag{111}$$

with $2n$ roots $x_1, x_2, \dots, x_{2n-1}$ and z such that $a_0 = z x_1 x_2 \cdots x_{2n-1}$. Thus $z = -y$ and consequently

$$y^{2n} + a_{n-1}y^{2n-2} + \dots + a_1 y^2 + a_0 = (b_0 + b_1 y^2 + \dots + b_{n-2} y^{2n-4})y,$$

relation equivalent to $\Delta y = 1$. Since the sum of the roots of (111) is 0, we have

$$y = x_1 + x_2 + \dots + x_{2n-1}$$

for $x_1, x_2, \dots, x_{2n-1}$ infinitely small.

It is also possible to take fx odd of degree $2n-1$ and φx even of degree $2n-2$. In this case, one finds that $\frac{1}{cy} = -\frac{a_0}{x_1 x_2 \dots x_{2n-1}}$.

When $\mu = 2n+1$ is odd, let us take

$$\begin{aligned}
&fx = (a_0 + a_1 x^2 + a_2 x^4 + \dots + a_{n-1}x^{2n-2} + x^{2n})x \quad \text{and}\\
&\varphi x = b_0 + b_1 x^2 + b_2 x^4 + \dots + b_{n-1}x^{2n-2},\\
&(fx)^2 - (\varphi x)^2(1-x^2)(1-c^2x^2) = (x^2 - x_1^2)(x^2 - x_2^2)\dots(x^2 - x_{2n}^2)(x^2 - y^2),\\
&fx_1 + \varphi x_1 \cdot \Delta x_1 = fx_2 + \varphi x_2 \cdot \Delta x_2 = \dots = fx_{2n} + \varphi x_{2n} \cdot \Delta x_{2n} = 0.
\end{aligned}$$

As in the preceding case, one obtains $y = \frac{b_0}{x_1 x_2 \dots x_{2n}}$ and $\Delta y = \frac{fy}{\varphi y}$. For $x_1, x_2, \dots, x_{2n}$ infinitesimal, $\Delta x_1, \Delta x_2, \dots, \Delta x_{2n}$ being 1, one has $y = x_1 + x_2 + \dots + x_{2n}$ and $\Delta y = 1$. One may also suppose fx even and φx odd, and then $\frac{1}{cy} = \frac{b_0}{x_1 x_2 \dots x_{2n}}$.

When $n = 1$, $fx = a_0 x + x^3$, $\varphi x = b_0$ where a_0 and b_0 are determined by the equations

$$a_0 x_1 + x_1^3 + b_0 \Delta x_1 = a_0 x_2 + x_2^3 + b_0 \Delta x_2 = 0$$

which give $a_0 = \frac{x_2^3 \Delta x_1 - x_1^3 \Delta x_2}{x_1 \Delta x_2 - x_2 \Delta x_1}$, $b_0 = \frac{x_2 x_1^3 - x_1 x_2^3}{x_1 \Delta x_2 - x_2 \Delta x_1}$. Then

$$y = \frac{b_0}{x_1 x_2} = \frac{x_1^2 - x_2^2}{x_1 \Delta x_2 - x_2 \Delta x_1} = \frac{x_1 \Delta x_2 + x_2 \Delta x_1}{1 - c^2 x_1^2 x_2^2}.$$

One may verify that $(a_0x+x^3)^2-b_0^2(1-x^2)(1-c^2x^2)=(x^2-x_1^2)(x^2-x_2^2)(x^2-y^2)$. The addition theorem takes the form $\varpi x_1+\varpi x_2=\varpi y+C$, $\varpi_0 x_0+\varpi_0 x_2=\varpi_0 y-x_1x_2y+C$,

$$\Pi x_1+\Pi x_2=\Pi y-\frac{a}{2\Delta a}\log\frac{a_0a+a^3+x_1x_2y\Delta a}{a_0a+a^3-x_1x_2y\Delta a}+C,$$

and $\Delta y=\frac{a_0y+y^3}{b_0}=\frac{a_0+y^2}{x_1x_2}$.

When $x_1,x_2,\dots,x_\mu=x$ ($\mu=2n-1$ or $2n$), the coefficients $a_0,a_1,\dots,b_0,b_1,\dots$ are determined by the equation $fx+\varphi x\cdot\Delta x=0$ and its first $\mu-1$ derivatives. Let $x_\mu=-\frac{a_0}{x^\mu}$ for $\mu=2n-1$, $\frac{b_0}{x^\mu}$ for $\mu=2n$ be the corresponding value of y, such that $\varpi x_\mu=C+\mu\varpi x$. One has $\varpi(x_{\mu+m})=C+\varpi x_\mu+\varpi x_m=C+\varpi y$ if $y=\frac{x_m\Delta x_\mu+x_\mu\Delta x_m}{1-c^2x_m^2x_\mu^2}$ and this equation gives $x_{\mu+m}=\frac{y\Delta e+e\Delta y}{1-c^2e^2y^2}$ where e is a constant. Letting x tend towards 0, one sees that $x_{\mu+m}$ is equivalent to $(m+\mu)x$ as is y, so that $e=0$, $\Delta e=1$ and

$$x_{\mu+m}=\frac{x_m\Delta x_\mu+x_\mu\Delta x_m}{1-c^2x_m^2x_\mu^2}. \tag{112}$$

In the same way, $x_{\mu-m}=\frac{x_m\Delta x_\mu-x_\mu\Delta x_m}{1-c^2x_m^2x_\mu^2}$. For $m=1$, this gives $x_{\mu+1}=-x_{\mu-1}+\frac{2x_\mu\Delta x}{1-c^2x^2x_\mu^2}$ and it is easy to deduce by induction that $x_{2\mu+1}$, $\frac{\Delta x_{2\mu+1}}{\Delta x}$, $\frac{x_{2\mu}}{\Delta x}$ and $\Delta x_{2\mu}$ are rational functions of x. One has $x_{\mu+m}x_{\mu-m}=\frac{x_\mu^2-x_m^2}{1-c^2x_\mu^2x_m^2}$; for $m=\mu-1$, this gives $x_{2\mu-1}=\frac{1}{x}\frac{x_\mu^2-x_{\mu-1}^2}{1-c^2x_\mu^2x_{\mu-1}^2}$. On the other hand, (112) with $m=\mu$ gives $x_{2\mu}=\frac{2x_\mu\Delta x_\mu}{1-c^2x_\mu^4}$.

Let us write $x_\mu=\frac{p_\mu}{q_\mu}$, $\Delta x_\mu=\frac{r_\mu}{q_\mu^2}$ where p_μ^2 and q_μ are polynomials in x without any common divisor. We have $\frac{p_{2\mu}}{q_{2\mu}}=\frac{2p_\mu q_\mu r_\mu}{q_\mu^4-c^2p_\mu^4}$ whence $p_{2\mu}=2p_\mu q_\mu r_\mu$, $q_{2\mu}=q_\mu^4-c^2p_\mu^4$, for these expressions are relatively prime. On the other hand, $\frac{xp_{2\mu-1}}{q_{2\mu-1}}=\frac{p_\mu^2q_{\mu-1}^2-q_\mu^2p_{\mu-1}^2}{q_\mu^2q_{\mu-1}^2-c^2p_\mu^2p_{\mu-1}^2}$ which is an irreducible fraction. Indeed the simultaneous equations $p_\mu^2q_{\mu-1}^2-q_\mu^2p_{\mu-1}^2=q_\mu^2q_{\mu-1}^2-c^2p_\mu^2p_{\mu-1}^2=0$ would give $x_\mu^2=x_{\mu-1}^2$ and $1-c^2x_\mu^2x_{\mu-1}^2=0$. Since $x_{2\mu-1}=\frac{x_\mu\Delta x_{\mu-1}+x_{\mu-1}\Delta x_\mu}{1-c^2x_\mu^2x_{\mu-1}^2}=\frac{x_\mu^2-x_{\mu-1}^2}{x_\mu\Delta x_{\mu-1}-x_{\mu-1}\Delta x_\mu}$, we should have $x_\mu\Delta x_{\mu-1}=x_{\mu-1}\Delta x_\mu=0$ and this is absurd for $x_\mu^2=\frac{1}{c}$. Thus $p_{2\mu-1}=\frac{1}{x}(p_\mu^2q_{\mu-1}^2-q_\mu^2p_{\mu-1}^2)$, $q_{2\mu-1}=q_\mu^2q_{\mu-1}^2-c^2p_\mu^2p_{\mu-1}^2$ and, from these relations, Abel recursively deduces that $p_{2\mu-1}$ is an odd polynomial in x of degree $(2\mu-1)^2$, $p_{2\mu}=p'\Delta x$ where p' is an odd polynomial of degree $(2\mu)^2-3$, q_μ is an even polynomial of degree μ^2-1 (resp. μ^2) when μ is odd (resp. even). More precisely,

$$p_{2\mu-1} = x(2\mu - 1 + A_2x^2 + \ldots + A_{(2\mu-1)^2-1}x^{(2\mu-1)^2-1}),$$
$$p_{2\mu} = x\Delta x(2\mu + B_2x^2 + \ldots + B_{4\mu^2-4}x^{(2\mu)^2-4}),$$
$$q_{2\mu-1} = 1 + A_4^1x^4 + \ldots + A^1_{(2\mu-1)^2-1}x^{(2\mu-1)^2-1},$$
$$q_{2\mu} = 1 + B_4^1x^4 + \ldots + B^1_{4\mu^2}x^{(2\mu)^2}.$$

In his second chapter (p. 545–557), Abel considers an exact differential form

$$y_1dx_1 + y_2dx_2 + \ldots + y_\mu dx_\mu \tag{113}$$

where the variables $x_1, x_2, \ldots, x_\mu$ are related by some algebraic relations in number less than μ and $y_1, y_2, \ldots, y_\mu$ are algebraic functions of them. He supposes that its primitive is of the form

$$\begin{aligned} u &+ A_1 \log v_1 + A_2 \log v_2 + \ldots + A_\nu \log v_\nu \\ &+ \alpha_1\psi t_1 + \alpha_2\psi t_2 + \ldots + \alpha_n\psi t_n \end{aligned} \tag{114}$$

where $A_1, A_2, \ldots, A_\nu, \alpha_1, \alpha_2, \ldots, \alpha_n$ are constants, $u, v_1, v_2, \ldots, v_\nu, t_1, t_2, \ldots, t_n$ algebraic functions of $x_1, x_2, \ldots, x_\mu$ and $\psi_m x = \int \frac{\theta' dx}{\Delta_m x}$ is an elliptic integral of modulus c_m $(1 \leq m \leq n)$, with $\Delta_m x = \pm\sqrt{(1-x^2)(1-c_m^2x^2)}$ and $\theta' = 1$, x^2 or $\frac{1}{1-\frac{x^2}{a^2}}$. Let us suppose that $x_1, x_2, \ldots, x_m$ are independent variables and that $x_{m+1}, x_{m+2}, \ldots, x_\mu$ are algebraic functions of them. Abel introduces an algebraic function θ such that

$$u, v_1, v_2, \ldots, v_\nu, t_1, t_2, \ldots, t_n, \Delta_1(t_1), \Delta_2(t_2), \ldots, \Delta_n(t_n) \tag{115}$$

are rationally expressible in $\theta, x_1, x_2, \ldots, x_\mu, y_1, y_2, \ldots, y_\mu$. He says that a convenient linear combination of the functions (115) has this property. In other words, Abel uses what is now called a Galois resolvant, which is most remarkable. Let $V = 0$ the minimal algebraic equation satisfied by θ, with coefficients rational with respect to $x_1, x_2, \ldots, x_\mu, y_1, y_2, \ldots, y_\mu$, and let δ be its degree. Writing that (113) is the differential of (114), one obtains a relation

$$p_1dx_1 + p_2dx_2 + \ldots + p_mdx_m = 0$$

where $p_1, p_2, \ldots, p_m$ are rational functions of $\theta, x_1, x_2, \ldots, x_\mu, y_1, y_2, \ldots, y_\mu$ and this implies that $p_1 = p_2 = \ldots = p_m = 0$. These last relations are still verified when θ is replaced by any one of the δ roots $\theta_1, \theta_2, \ldots, \theta_\delta$ of $V = 0$ because it is an irreducible equation. It results that

$$\begin{aligned} \delta \textstyle\int(y_1dx_1 + y_2dx_2 + \ldots + y_\mu dx_\mu) = U &+ A_1 \log V_1 + \ldots + A_\nu \log V_\nu \\ &+ \alpha_1(\psi_1t_1' + \psi_1t_1'' + \ldots + \psi_1t_1^{(\delta)}) + \ldots \\ &+ \alpha_n(\psi_nt_n' + \psi_nt_n'' + \ldots + \psi_nt_n(\delta)) \end{aligned}$$

where $U = u' + u'' + \ldots + u^{(\delta)}$ is the sum of the values taken by u when θ is successively replaced by $\theta_1, \theta_2, \ldots, \theta_\delta$, $\log V_m = \log v_m' + \log v_m'' + \ldots + \log v_m^{(\delta)}$ is the analogous sum associated to $\log v_m$ and $t_m', t_m'', \ldots, t_m^{(\delta)}$ are the values taken by t_m. Now, by the addition theorem for elliptic integrals, $\psi_m t'_m + \psi_m t''_m + \ldots + \psi_m t_m(\delta) = \psi_m T_m + p + B_1 \log q_1 + B_2 \log q_2 + \ldots + B_\nu \log q_\nu$ where T_m, $\Delta_m T_m$, p, $q_1, q_2, \ldots, q_\nu$ are rational functions of $t'_m, t''_m, \ldots, t_m^{(\delta)}, \Delta_m(t'_m), \Delta_m(t''_m), \ldots, \Delta_m(t_m^{(\delta)})$ and consequently of $\theta_1, \theta_2, \ldots, \theta_\delta, x_1, x_2, \ldots, x_\mu, y_1, y_2, \ldots, y_\mu$. But since they are symmetric with respect to $\theta_1, \theta_2, \ldots, \theta_\delta$, they are rational functions of $x_1, x_2, \ldots, x_\mu, y_1, y_2, \ldots, y_\mu$ as are U and the V_m. We finally obtain a relation of the form

$$\begin{aligned}
&\delta \textstyle\int (y_1 dx_1 + y_2 dx_2 + \ldots + y_\mu dx_\mu \\
&= r + A' \log \rho' + A'' \log \rho'' + \ldots + A^{(k)} \log \rho^{(k)} \\
&\quad + \alpha_1 \psi_1 \theta_1 + \alpha_2 \psi_2 \theta_2 + \ldots + \alpha_n \psi_n \theta_n
\end{aligned}$$

where δ is an integer, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the same as in (114), $A', A'', \ldots$ are constants and $\theta_1, \Delta_1(\theta_1), \theta_2, \Delta_2(\theta_2), \ldots, \theta_n, \Delta_n(\theta_n), r, \rho', \rho'', \ldots, \rho^{(k)}$ are *rational* functions of $x_1, x_2, \ldots, x_\mu, y_1, y_2, \ldots, y_\mu$.

A particular case concerns the differential forms (113) of which the primitive is of the form $u + A_1 \log v_1 + A_2 \log v_2 + \ldots + A_\nu \log v_\nu$ where $u, v_1, v_2, \ldots, v_\nu$ are *algebraic* functions of $x_1, x_2, \ldots, x_\mu$. Then one may suppose that $u, v_1, v_2, \ldots, v_\nu$ are *rational* functions of $x_1, x_2, \ldots, x_\mu, y_1, y_2, \ldots, y_\mu$. In a footnote, Abel announces a general theory, based on this result, for the reduction of integrals of algebraic differential forms by algebraic and logarithmic functions.

Applied to elliptic integrals, the preceding theorem takes the form: if

$$\begin{aligned}
&\int \left(\frac{\alpha_1 r_1}{\Delta_1 x_1} dx_1 + \frac{\alpha_2 r_2}{\Delta_2 x_2} dx_2 + \ldots + \frac{\alpha_\mu r_\mu}{\Delta_\mu x_\mu} dx_\mu \right) \\
&= u + A_1 \log v_1 + A_2 \log v_2 + \ldots + A_\nu \log v_\nu \qquad (116)
\end{aligned}$$

where $r_1, r_2, \ldots, r_\mu$ are rational functions and $u, v_1, v_2, \ldots, v_\nu$ are *algebraic* functions of $x_1, x_2, \ldots, x_\mu$, one may suppose that $u, v_1, v_2, \ldots, v_\nu$ are rational functions of $x_1, x_2, \ldots, x_\mu, \Delta_1 x_1, \Delta_2 x_2, \ldots, \Delta_\mu x_\mu$. From (116), we may also conclude that there exists an integer δ such that

$$\begin{aligned}
&\delta \alpha_1 \psi_1 x_1 + \delta \alpha_2 \psi_2 x_2 + \ldots + \delta \alpha_m \psi_m x_m + \alpha_{m+1} \psi_{m+1} \theta_1 + \ldots + \alpha_\mu \psi_\mu \theta_{\mu-m} \\
&= r + A' \log \rho' + A'' \log \rho'' + \ldots + A^{(k)} \log \rho^{(k)}
\end{aligned}$$

where $\psi_j x = \int \frac{r_1}{\Delta_j x} dx$ and $\theta_1, \Delta_{m+1}\theta_1, \theta_2, \Delta_{m+2}\theta_2, \ldots, \theta_{\mu-m}, \Delta_\mu \theta_{\mu-m}, r, \rho', \rho'', \ldots, \rho^{(k)}$ are rational functions of $x_1, x_2, \ldots, x_m, \Delta_1 x_1, \Delta_2 x_2, \ldots, \Delta_m x_m$. When only one elliptic integral $\psi_m x$ is isolated, this gives

$$\begin{aligned}
\delta \alpha_m \psi_m x = &-\alpha_1 \psi_1 \theta_1 - \alpha_2 \psi_2 \theta_2 - \ldots - \alpha_{m-1} \psi_{m-1} \theta_{m-1} - \alpha_{m+1} \psi_{m+1} \theta_{m+1} - \ldots \\
&-\alpha_\mu \psi_\mu \theta_\mu + r + A' \log \rho' + A'' \log \rho'' + \ldots + A^{(k)} \log \rho^{(k)} \qquad (117)
\end{aligned}$$

where $\theta_1, \Delta_{m+1}\theta_1, \theta_2, \Delta_{m+2}\theta_2, \ldots, r, \rho', \rho'', \ldots$ are rational functions of x and $\Delta_m x$, that is of the form $p + q\Delta_m x$ with p, q rational in x.

When $x_1 = x_2 = \ldots = x_\mu = x$ and $c_1 = c_2 = \ldots = c_\mu = c$, one obtains the following theorem: if there is a relation

$$\alpha\varpi x + \alpha_0\varpi_0 x + \alpha_1\Pi_1 x + \alpha_2\Pi_2 x + \ldots + \alpha_\mu\Pi_\mu x$$
$$= u + A_1 \log v_1 + A_2 \log v_2 + \ldots + A_\nu \log v_\nu$$

where $u, v_1, v_2, \ldots, v_\nu$ are algebraic functions of x, one may suppose that they are of the form $p + q\Delta x$ with p, q rational in x.

Differentiating (117) we obtain a relation of the form $P + Q\Delta_m x = 0$ which implies $P = Q = 0$ and therefore $P - Q\Delta_m x = 0$. When the sign of $\Delta_m x$ is changed into the opposite, the θ_j take new values θ'_j and we have $-\delta\alpha_m\psi_m x = -\sum\alpha\psi\theta' + v'$ where v' designates the algebraic and logarithmic part. It results that $2\delta\alpha_m\psi_m x = \sum\alpha(\psi\theta' - \psi\theta) + v - v'$ where, by the addition theorem,

$$\psi\theta' - \psi\theta = \psi y - v''$$

if $y = \frac{\theta'\Delta\theta - \theta\Delta\theta'}{1-c^2\theta^2\theta'^2}$, v'' denoting an algebraic and logarithmic function. Now $\theta = p + q\Delta_m x$ and $\Delta\theta = r + \rho\Delta_m x$ where p, q, r, ρ are rational functions of x and it results that $\theta' = p - q\Delta_m x$, $\Delta\theta' = r - \rho\Delta_m x$ and that $y = t\Delta_m x$ where t is a rational function of x. Then it is easy to see that Δy is a rational function of x. One may replace y by $z = \frac{y\Delta e + e\Delta y}{1-c^2e^2y^2}$ where e is a constant because ψy and ψz differ by an algebraic and logarithmic function. For $e = 1$, $z = \frac{\Delta y}{1-c^2y^2}$ is a rational function of x and $\Delta z = \frac{c^2-1}{1-c^2y^2}y$ has a rational ratio to $\Delta_m x$. We have $2\delta\alpha_m\psi_m x = \sum\alpha\psi z + V$ where V is an algebraic and logarithmic function. Then $V = u + A_1 \log v_1 + A_2 \log v_2 + \ldots$ where $u, v_1, v_2, \ldots$ are of the form $p + q\Delta_m x$ with p and q rational in x.

Taking $m = \mu$, we obtain $2\delta\alpha_\mu\psi_\mu x_\mu = \alpha_1\psi_1 z_1 + \alpha_2\psi_2 z_2 + \ldots + \alpha_{\mu-1}\psi_{\mu-1}z_{\mu-1} + V$ and we may eliminate $\psi_\mu x_\mu$ between this relation and (116), getting

$$\alpha_1(2\delta\psi_1 x_1 - \psi_1 z_1) + \ldots + \alpha_{\mu-1}(2\delta\psi_{\mu-1}x_{\mu-1} - \psi_{\mu-1}z_{\mu-1}) = V'.$$

Since δ is an integer, there exist $x'_1, x'_2, \ldots, x'_{\mu-1}$ such that $2\delta\psi_j x_j - \psi_j z_j = \psi_j x'_j + V_j$ $(1 \le j \le \mu - 1)$ and we have $\alpha_1\psi_1 x'_1 + \alpha_2\psi_2 x'_2 + \ldots + \alpha_{\mu-1}\psi_{\mu-1}x'_{\mu-1} = u' + A'_1 \log v'_1 + A'_2 \log v'_2 + \ldots + A'_{\nu'} \log v'_{\nu'}$ of the same form as (116) with one elliptic integral less. We may iterate until we arrive at a relation with only algebraic and logarithmic functions.

The general problem (106) has been reduced to the following one: "To satisfy in the most general manner the equation

$$\psi x = \beta_1\psi_1 y_1 + \beta_2\psi_2 y_2 + \ldots + \beta_n\psi_n y_n$$
$$+ u + A_1 \log v_1 + A_2 \log v_2 + \ldots + A_\nu \log v_\nu \tag{118}$$

where $\psi, \psi_1, \psi_2, \ldots, \psi_n$ designate elliptic integrals of the three kinds, supposing that $y_1, y_2, \ldots, y_n$ are rational functions of x and that $\Delta_1 y_1, \Delta_2 y_2, \ldots, \Delta_n y_n$ are of the form $p\Delta x$ where p is rational in x and Δx designate the radical which appears in the function ψx." If $\Delta_m y_m = p_m\Delta x$ and $\psi_m x = \int \frac{\theta_m x \cdot dx}{\Delta_m x}$ where $\theta_m x$ is rational, we have $\psi_m y_m = \int \frac{\theta_m y_m}{p_m}\frac{dy_m}{dx}\frac{dx}{\Delta x}$ where $\frac{\theta_m y_m}{p_m}\frac{dy_m}{dx}$ is a rational function of x. Thus $\psi_m y_m = r + A\varpi x + A_0\varpi_0 x + A'\Pi(x, a') + A''\Pi(x, a'') + \ldots$ where r is an algebraic and logarithmic expression. Equation (118) finally takes the form

$$\alpha\varpi x + \alpha_0\varpi_0 x + \alpha_1 \Pi(x, a_1) + \alpha_2 \Pi(x, a_2) + \ldots + \alpha_\mu \Pi(x, a_\mu)$$
$$= u + A_1 \log v_1 + A_2 \log v_2 + \ldots \qquad (119)$$

The problem (118) is thus reduced to the three following ones:

A) To find all the possible cases in which

$$(1 - y^2)(1 - c'^2 y^2) = p^2(1 - x^2)(1 - c^2 x^2)$$

with y and p rational functions of x (c, c' are constants).

B) To reduce $\varpi(y, c')$, $\varpi_0(y, c')$ and $\Pi(y, c', a)$, where y and c' are as in A), to the form

$$r + A\varpi x + A_0\varpi_0 x + A'\Pi(x, a') + A''\Pi(x, a'') + \ldots$$

C) To find the necessary and sufficient conditions for (119) to be satisfied.

The third chapter (p. 557–565) is devoted to the solution of problem C), where one may suppose that $u, v_1, v_2, \ldots, v_\nu$ are of the form $p + q\Delta x$, p and q rational in x. Abel takes the problems dealt with in the second chapter of his unpublished memoir *Théorie des transcendantes elliptiques* (see our §4) in a more general setting. Equation (119) is rewritten

$$\psi x = u + \sum A \log v,$$

where $\psi x = \beta\varpi x + \beta_0\varpi_0 x + \beta_1 \Pi\alpha_1 + \beta_2 \Pi\alpha_2 + \ldots + \beta_n \Pi\alpha_n$ and $\Pi\alpha_m = \int \frac{dx}{\left(1 - \frac{x^2}{\alpha_m^2}\right)\Delta x}$; it is supposed that it is impossible to find any similar relation not containing all the $\Pi\alpha_m$ and that all the α_m are different from ± 1 and $\pm\frac{1}{c}$. Changing the sign of Δx, we obtain $-\psi x = u' + \sum A \log v'$ and then $2\psi x = u - u' + \sum A \log \frac{v}{v'}$. Changing the sign of x without changing that of Δx, we obtain $-2\psi x = u'' - u''' + \sum A \log \frac{v''}{v'''}$ and

$$\psi x = \frac{1}{4}(u - u' - u'' + u''') + \frac{1}{4}\sum A \log \frac{vv'''}{v'v''}.$$

If $v = p + qx + (p' + q'x)\Delta x$ where p, q, p', q' are even functions, $v' = (p + qx) - (p' + q'x)\Delta x$, $v'' = (p - qx) + (p' - q'x)\Delta x$ and $v''' = p - qx - (p' - q'x)\Delta x$. Thus $\frac{vv'''}{v'v''} = \frac{fx + \varphi x \cdot \Delta x}{fx - \varphi x \cdot \Delta x}$ where fx and φx are polynomials, one even and the other one odd. The algebraic part $\frac{1}{4}(u - u' - u'' + u''')$ is of the form $r\Delta x$ where r is an odd rational function of x and we may rewrite our equation in the form

$$\psi x = r\Delta x + \sum A \log \frac{fx + \varphi x \cdot \Delta x}{fx - \varphi x \cdot \Delta x} \qquad (120)$$

with A in place of $\frac{1}{4}A$. We may suppose that there is no linear relation with integer coefficients between the A_m, otherwise it would be possible to reduce the number ν of the terms in the sum.

Differentiating one term $\rho = \log \frac{fx+\varphi x\cdot\Delta x}{fx-\varphi x\cdot\Delta x}$, we obtain $d\rho = \frac{vdx}{\theta x\cdot\Delta x}$ where

$$\theta x = (fx)^2 - (\varphi x)^2(\Delta x)^2 \quad \text{and} \quad v\varphi x = 2f'x\theta x - fx\theta' x,$$

so that v is an even polynomial. If the roots of θx are $\pm a_1, \pm a_2, \ldots, \pm a_\mu$, the decomposition of $\frac{v}{\theta x}$ in simple elements is of the form $k + \frac{\beta_1'}{a_1^2-x^2} + \frac{\beta_2'}{a_2^2-x^2} + \ldots + \frac{\beta_\mu'}{a_\mu^2-x^2}$ where k is a constant and $\beta_j' = 2m_j a_j \frac{fa_j}{\varphi a_j} = -2m_j a_j \Delta a_j$ where m_j is the multiplicity of a_j as a root of θx. Thus the differentiation of (120) gives, after multiplication by Δx:

$$\begin{aligned}
&\beta + \beta_0 x^2 + \frac{\alpha_1^2\beta_1}{a_1^2-x^2} + \frac{\alpha_2^2\beta_2}{a_2^2-x^2} + \ldots + \frac{\alpha_n^2\beta_n}{a_n^2-x^2} \\
&= \frac{dr}{dx}(\Delta x)^2 - r((1+c^2)x - 2c^2x^3) \\
&\quad + A_1\left(k_1 - \frac{2m_1 a_1 \Delta a_1}{a_1^2-x^2} - \frac{2m_2 a_2 \Delta a_2}{a_2^2-x^2} - \ldots\right) \\
&\quad + A_2\left(k_2 - \frac{2m_1' a_1' \Delta a_1'}{a_1'2-x^2} - \frac{2m_2' a_2' \Delta a_2'}{a_2'2-x^2} - \ldots\right) + \ldots .
\end{aligned}$$

From this relation, Abel deduces that $r = 0$ and that only one of the coefficients A_m may be different from 0. He takes $A_1 = 1$, $A_2 = A_3 = \ldots = A_\nu = 0$ and finds $\beta = k_1$, $\beta_0 = 0$, $\alpha_1 = a_1$, $\alpha_2 = a_2, \ldots$, $\beta_1 = -\frac{2m_1\Delta a_1}{a_1}$, $\beta_2 = -\frac{2m_2\Delta a_2}{a_2}, \ldots$. The most general relation between elliptic integrals with the same modulus is thus of the form

$$\begin{aligned}
&\beta\varpi x - \frac{2m_1\Delta\alpha_1}{\alpha_1}\Pi\alpha_1 - \ldots - \frac{2m_n\Delta\alpha_n}{\alpha_n}\Pi\alpha_n \\
&= \log\frac{fx+\varphi x\cdot\Delta x}{fx-\varphi x\cdot\Delta x} + C
\end{aligned} \tag{121}$$

where the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ are related by the equation

$$(fx)^2 - (\varphi x)^2(1-x^2)(1-c^2x^2) = (x^2-\alpha_1^2)^{m_1}(x^2-\alpha_2^2)^{m_2}\ldots(x^2-\alpha_n^2)^{m_n}. \tag{122}$$

Abel remarks that this implies

$$\begin{aligned}
&m_1\varpi\alpha_1 + m_2\varpi\alpha_2 + \ldots + m_n\varpi\alpha_n = C \quad \text{and} \\
&m_1\Pi'\alpha_1 + m_2\Pi'\alpha_2 + \ldots + m_n\Pi'\alpha_n = C - \frac{a}{2\Delta a}\log\frac{fa+\varphi a\cdot\Delta a}{fa-\varphi a\cdot\Delta a}
\end{aligned}$$

if $\Pi'\alpha = \int \frac{d\alpha}{\left(1-\frac{\alpha^2}{a^2}\right)\Delta\alpha}$.

When $n = 1$, $\alpha_1 = \alpha$ and $m_1 = m$ we have $\Pi\alpha = \frac{\beta\alpha}{2m\Delta\alpha}\varpi x - \frac{\alpha}{2m\Delta\alpha}\log\frac{fx+\varphi x\cdot\Delta x}{fx-\varphi x\cdot\Delta x}$ if the parameter α verifies

$$(fx)^2-(\varphi x)^2(1-x^2)(1-c^2x^2)=(x^2-\alpha^2)^m. \tag{123}$$

For $m=2$, $fx=ax$ is of degree 1, $\varphi x=\frac{1}{c}\sqrt{-1}$ is constant and $(x^2-\alpha^2)^2=x^4-\left(\frac{1+c^2}{c^2}-a^2\right)x^2+\frac{1}{c^2}$ and this gives $a=1\mp\frac{1}{c}$, $\alpha=\frac{1}{\sqrt{\pm c}}$. For $m=3$ and fx odd, $fx=x^3+ax$, $\varphi x=b$ and

$$(x^3+ax)^2-b^2(1-x^2)(1-c^2x^2)=(x^2-\alpha^2)^3,$$

equation which leads to $\alpha^3=b$, $\alpha^3+a\alpha+b\Delta\alpha=0$, $2a-c^2b^2=-3\alpha^2$, $a^2+(1+c^2)b^2=3\alpha^4$. This gives $a=\frac{1}{2}(c^2\alpha^6-3\alpha^2)$ and α determined by the equation $\Delta\alpha=\frac{1}{2}(1-c^2\alpha^4)$.

Generally, as Abel states, α must be a zero or a pole of the function x_m defined in the first chapter and such that $\frac{dx_m}{\Delta x_m}=m\frac{dx}{\Delta x}$ ($x_m=0$ for $x=0$). Indeed we have

$$p^2-q^2(\Delta x)^2=(x^2-\alpha^2)^m(x^2-\alpha_m)$$

where α_m is the value of x_m for $x=\alpha$ (chapter I) and, multiplying by (123), we obtain

$$\begin{aligned}(p\,fx\pm q\varphi x(\Delta x)^2)^2-(p\varphi x\pm q\,fx)^2(\Delta x)^2&=p^2(fx)^2-q^2(\varphi x)^2(\Delta x)^4\\&=(x^2-\alpha^2)^{2m}(x^2-\alpha_m).\end{aligned}$$

It results that $p\,fx+q\varphi x(\Delta x)^2$ or $p\,fx-q\varphi x(\Delta x)^2$ is divisible by $(x^2-\alpha^2)^m$ and we have a relation $r^2-\rho^2(\Delta x)^2=x^2-\alpha_m^2$ where r, ρ are polynomials, one even and the other odd. But this relation implies that $\rho=0$ and that $\alpha_m=0$ or $\frac{1}{0}$. Conversely, it is easy to see that such an α satisfies an equation (123). Abel remarks that, in these cases, the coefficient β in (121) is always different from 0, so that there is no elliptic differential of the third kind integrable by algebraic and logarithmic functions.

When $n=3$ and $m_1=m_2=m_3=1$, (121) takes the form

$$\frac{\Delta\alpha_1}{\alpha_1}\Pi\alpha_1+\frac{\Delta\alpha_2}{\alpha_2}\Pi\alpha_2=\frac{\Delta\alpha}{\alpha}\Pi\alpha+\beta\varpi x-\frac{1}{2}\log\frac{fx+\varphi x\cdot\Delta x}{fx-\varphi x\cdot\Delta x}$$

where $fx=x^3+ax$, $\varphi x=b$ and $(x^3+ax)^2-b^2(1-x^2)(1-c^2x^2)=(x^2-\alpha^2)\times(x^2-\alpha_1^2)(x^2-\alpha_2^2)$. This gives $\alpha=\frac{\alpha_1\Delta\alpha_2+\alpha_2\Delta\alpha_1}{1-c^2\alpha_1^2\alpha_2^2}$, $b=\alpha\alpha_1\alpha_2$,

$$a=\frac{1}{2}(c^2\alpha^2\alpha_1^2\alpha_2^2-\alpha^2-\alpha_1^2-\alpha_2^2),\quad \frac{\Delta\alpha}{\alpha}=\frac{\alpha^2+a}{\alpha\alpha_1\alpha_2},\quad \beta=-c^2\alpha\alpha_1\alpha_2$$

(cf. chapter I). In particular, for α_2 infinite, $\alpha=\pm\frac{1}{c\alpha_1}$ and $\Pi\alpha+\Pi\left(\frac{1}{c\alpha}\right)=\varpi x+\frac{1}{2}\frac{\alpha}{\Delta\alpha}\log\frac{x\Delta\alpha+\alpha\Delta x}{x\Delta\alpha-\alpha\Delta x}$. Other relations between two elliptic integrals of the third kind are obtained by (121) with $n=2$.

In the fourth chapter (p. 565–606), Abel solves the problem A) of the second chapter, that is to satisfy the equation $(1-y^2)(1-c'^2y^2)=r^2(1-x^2)(1-c^2x^2)$, y and r being rational functions of x. Since $1-y^2$ and $1-c'^2y^2$ have no common factor ($c' \neq 1$), this equation implies $1 - y^2 = r_1^2 \rho$, $1 - c'^2 y^2 = r_2^2 \rho'$ where r_1, r_2 are rational functions of x, $r_1 r_2 = r$ and $\rho\rho' = (1 - x^2)(1 - c^2 x^2)$. Differentiating, we obtain $-2ydy = r_1(r_1 d\rho + 2\rho dr_1)$, $-2c'^2 ydy = r_2(r_2 d\rho' + 2\rho' dr_2)$ which show that the numerator of $\frac{dy}{dx}$ is divisible by r_1 and r_2, and so by their product r: $\frac{dy}{dx} = rv$ where v is a rational function without any pole among the zeros of r. Let $y = \frac{p}{q}$, irreducible fraction where p, q are polynomials of respective degrees m, n. One has $r = \frac{\theta}{q^2}$ where θ is a polynomial and $\theta v = q^2 \frac{dy}{dx} = \frac{qdp - pdq}{dx}$, whence v is a polynomial. If $m > n$, the equation

$$(q^2 - p^2)(q^2 - c'^2 p^2) = \theta^2 (1 - x^2)(1 - c^2 x^2)$$

shows that $4m = 2\mu + 4$ where μ is the degree of θ. If ν is the degree of v, we then see that $\mu + \nu = m + n - 1$ and $\nu = m + n - 1 - \nu < 2m - \mu - 1 = 1$. Thus $\nu = 0$ and v is constant. In the same way, if $n > m$, we have $4n = 2\mu + 4$, $\nu < 2n - \mu - 1 = 1$ and $\nu = 0$. In the remaining case, in which $m = n$, it is for instance possible that $q - p = \varphi$ be of degree $m - k < m$. Then $4m - k = 2\mu + 4$ and $\mu + \nu = 2m - k - 1$ for $\theta v = \frac{pd\varphi - \varphi dp}{dx}$ and $\nu = 2m - k - 1 - \mu = 1 - \frac{1}{2}k$ is again 0. In any case v is a constant ε and

$$\frac{dy}{\sqrt{(1 - y^2)(1 - c'^2 y_2)}} = \frac{\varepsilon dx}{\sqrt{(1 - x^2)(1 - c^2 x^2)}}. \tag{124}$$

The second result announced in the introduction is thus demonstrated.

It remains to determine the rational function y and the transformed modulus c'. Abel begins by considering the case in which $y = \frac{\alpha + \beta x}{\alpha' + \beta' x}$ and he explains the 6 cases already met in *Sur le nombre de transformations différentes* ... (our §7). He then considers the case in which $y = \psi x = \frac{A_0 + A_1 x + A_2 x^2 + \ldots + A_\mu x^\mu}{B_0 + B_1 x + B_2 x^2 + \ldots + B_\mu x^\mu}$ (irreducible fraction, one of the coefficients A_μ, B_μ different from 0). The treatment uses only the addition theorem for elliptic integrals of chapter I and not the elliptic function λ and its double periodicity as in the preceding memoirs; but the lines are similar. If x, x' are two roots of the equation $y = \psi x$, one has $\frac{dx}{\Delta x} = \frac{1}{\varepsilon} \frac{dy}{\Delta' y} = \frac{dx\prime}{\Delta x\prime}$ and consequently $x' = \frac{x \Delta e + e \Delta x}{1 - c^2 e^2 x^2} = \theta x$ where e is a constant. Thus $\psi(\theta x) = \psi x$ and we see that the equation $y = \psi x$ has the roots $x, \theta x, \theta^2 x, \ldots, \theta^n x, \ldots$ where it is easy to see that

$$\theta^n x = \frac{x \Delta e_n + e_n \Delta x}{1 - c^2 e_n^2 x^2},$$

e_n being the rational function of e defined by $\frac{de_n}{\Delta e_n} = n \frac{de}{\Delta e}$ and $e_n = 0$ for $e = 0$ (see chapter I). Since the equation has only μ roots, there exists an n such that $\theta^n x = x$ that is $e_n = 0$ and $\Delta e_n = 1$. These equations are equivalent to $\frac{\Delta e_n}{1 - c^2 e_n^2} = 1$, which is of degree n^2 in e. The number n must be minimal and we must eliminate the roots e which would lead to $e_\mu = 0$, $\Delta \varepsilon_\mu = 1$ for a $\mu < n$. If, for instance, n is a prime number, the root $e = 0$ is to be eliminated and it remains $n^2 - 1$ solutions e.

Let us suppose that two rational functions $\psi z = \frac{p}{q}$, $\psi' z = \frac{p'}{q'}$ where p, q, p', q' are polynomials of degree μ and the two fractions are irreducible. If the equations

$y = \psi(x)$ and $y' = \psi'(x)$ have the same roots $x, x', x'', \ldots, x^{(\mu-1)}$ we have $\frac{p-qy}{p'-q'y'} = \frac{a-by}{a'-b'y'}$ where a, b, a', b' are the respective coefficients of z^μ in p, q, p', q' and z has any value. We draw $y' = \frac{\alpha+\beta y}{\alpha'+\beta' y}$; if moreover y and y' correspond to the same modulus c', we have $y' = \frac{1}{c'y}$.

When $n = \mu$, the roots of $y = \psi x$ are $x, \theta x, \ldots, \theta^{n-1}x$ and

$$p - qy = (a - by)(z - x)(z - \theta x) \cdots (z - \theta^{n-1}x). \tag{125}$$

We can draw y from this equation, giving to z a particular value. If n is odd, noted $2\mu + 1$, putting $z = 0$, we obtain $y = \frac{a'+ax\cdot\theta x\cdot\theta^2 x\cdots\theta^{2\mu}x}{b'+bx\cdot\theta x\cdot\theta^2 x\cdots\theta^{2\mu}x}$ where a', b' are the respective constant terms of p, q. Since $e_{n-m} = -e_m$ and $\Delta e_{n-m} = \Delta e_m$, we see that $\theta^{n-m}x = \frac{x\Delta e_m - e_m\Delta x}{1-c^2e_m^2x^2}$ and $\theta^m x\cdot\theta^{n-m}x = \frac{x^2-e_m^2}{1-c^2e_m^2x^2}$. It results that the value found for y is rational in x. Moreover, it is invariant by the substitution $x \mapsto \theta x$ because $\theta^{2\mu+1}x = x$, and it results that (125) is verified for any value of z. For $x = \pm 1$ or $\pm\frac{1}{c}$, $\Delta x = 0$ and $\theta^m x = \theta^{2\mu+1-m}x$, so that

$$p - q\alpha = (a - b\alpha)(1 - z)\rho^2, \quad p - q\beta = (a - b\beta)(1 + z)\rho'^2,$$
$$p - q\gamma = (a - b\gamma)(1 - cz)\rho''^2, \quad p - q\delta = (a - b\delta)(1 + cz)\rho'''^2$$

where $\alpha, \beta, \gamma, \delta$ are the values of y corresponding to $x = 1, -1, \frac{1}{c}, -\frac{1}{c}$ and $\rho, \rho', \rho'', \rho'''$ are polynomials of degree μ in z. Now we want that

$$(q^2 - p^2)(q^2 - c'^2p^2) = r^2(1 - z^2)(1 - c^2z^2)$$

and this implies that $\{\alpha, \beta, \gamma, \delta\} = \left\{1, -1, \frac{1}{c'}, -\frac{1}{c'}\right\}$; conversely, this condition will be sufficient. Let us take $\alpha = 1$, $\beta = -1$, $\gamma = \frac{1}{c'}$, $\delta = -\frac{1}{c'}$. Since $y = \frac{a'+a\varphi x}{b'+b\varphi x}$ where

$$\varphi x = x\cdot\theta x\cdot\theta^2x\cdots\theta^{2\mu}x = \frac{x(x^2 - e^2)(x^2 - e_2^2)\cdots(x^2 - e_\mu^2)}{(1 - c^2e^2x^2)(1 - c^2e_2^2x^2)\cdots(1 - c^2e_\mu^2x^2)}$$

is an odd function, we have $\alpha = \frac{a'+a\varphi(1)}{b'+b\varphi(1)}$, $\beta = \frac{a'-a\varphi(1)}{b'-b\varphi(1)}$, $\gamma = \frac{a'+a\varphi\left(\frac{1}{c}\right)}{b'+b\varphi\left(\frac{1}{c}\right)}$, $\delta = \frac{a'-a\varphi\left(\frac{1}{c}\right)}{b'-b\varphi\left(\frac{1}{c}\right)}$ or $a' \mp b' \pm (a \mp b)\varphi(1) = 0$, $a' \mp \frac{b'}{c'} \pm \left(a \mp \frac{b}{c'}\right)\varphi\left(\frac{1}{c}\right) = 0$. These equations are compatible only if a' or b' is 0 ($c' \neq 1$). Let us suppose that $a' = 0 = b$; we have $c' = \frac{\varphi(1)}{\varphi\left(\frac{1}{c}\right)}$, $y = \frac{a}{b'}\varphi x = \frac{\varphi x}{\varphi(1)}$ where $\varphi(1) = \frac{1-e^2}{1-c^2e^2}\frac{1-e_2^2}{1-c^2e_2^2}\cdots\frac{1-e_\mu^2}{1-c^2e_\mu^2}$, $\varphi\left(\frac{1}{c}\right) = \frac{1}{c^{2\mu+1}}\frac{1-c^2e^2}{1-e^2}\frac{1-c^2e_2^2}{1-e_2^2}\cdots\frac{1-c^2e_\mu^2}{1-e_\mu^2} = \frac{1}{c^{2\mu+1}\varphi(1)}$. Then $c' = c^{2\mu+1}(\varphi(1))^2$. In order to determine the multiplicator ε, Abel uses the value of $\frac{dy}{dx} = \varepsilon\frac{\Delta' y}{\Delta x}$ for $x = 0$, which is $\pm e^2e_2^2\cdots e_\mu^2\frac{1}{\varphi(1)}$; thus $\varepsilon = e^2e_2^2\cdots e_\mu^2\frac{c^{\mu+\frac{1}{2}}}{\sqrt{c'}}$. He has reconstituted the formulae for the transformations of odd order $2\mu + 1$: if e is a root of the equation $e_{2\mu+1} = 0$ which does not satisfy any other equation $e_{2m+1} = 0$ where $2m + 1$ is a divisor of $2\mu + 1$, let us put

$$y = \frac{c^{\mu+\frac{1}{2}}}{\sqrt{c'}} \frac{x(e^2-x^2)(e_2^2-x^2)\cdots(e_\mu^2-x^2)}{(1-c^2e^2x^2)(1-c^2e_2^2x^2)\cdots(1-c^2e_\mu^2x^2)},$$
$$c' = c^{2\mu+1}\left(\frac{(1-e^2)(1-e_2^2)\cdots(1-e_\mu^2)}{(1-c^2e^2)(1-c^2e_2^2)\cdots(1-c^2e_\mu^2)}\right)^2,$$
$$\varepsilon = \frac{c^{\mu+\frac{1}{2}}}{\sqrt{c'}} e^2e_2^2\dots e_\mu^2. \tag{126}$$

Then we have $\frac{dy}{\sqrt{(1-y^2)(1-c'^2y^2)}} = \pm\varepsilon\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$. Five other systems (y, c', ε) corresponding to the same value of e are obtained by composing with a transformation of order 1.

For instance, when $\mu = 1$, $2\mu+1 = 3$ is prime and we may take for e any root different from 0 of the equation $e_3 = 0$, that is $0 = 3 - 4(1+c^2)e^2 + 6c^2e^4 - c^4e^8$ of degree 4 in e^2 and the $c' = c^3\left(\frac{1-e^2}{1-c^2e^2}\right)^2$, $\varepsilon = c\sqrt{\frac{c}{c'}}e^2$, $y = \frac{c\sqrt{c}}{\sqrt{c'}}\frac{x(e^2-x^2)}{1-c^2e^2x^2}$. Eliminating e, we obtain the modular equation in the form

$$(c'-c)^2 = 4\sqrt{cc'}(1-\sqrt{cc'})^2.$$

The roots of the equation $0 = \frac{c^{\mu+\frac{1}{2}}}{\sqrt{c'}}z(z-e^2)(z-e_2^2)\cdots(z-e_\mu^2) + y(1-c^2e^2z^2)(1-c^2e_2^2z^2)\cdots(1-c^2e_\mu^2z^2)$ are $x, \theta x, \dots, \theta^{2\mu}x$, thus $x + \theta x + \dots + \theta^{2\mu}x = \frac{(-1)^{\mu+1}c^{2\mu}e^2e_2^2\cdots e_\mu^2}{c^{\mu+\frac{1}{2}}c'^{-\frac{1}{2}}}y$. Since $\theta^m x + \theta^{2\mu+1-m}x = \frac{2\Delta e_m x}{1-c^2e_m^2x^2}$, this gives $y = \left(x + \frac{2\Delta e\cdot x}{1-c^2e^2x^2} + \frac{2\Delta e_2\cdot x}{1-c^2e_2^2x^2} + \dots + \frac{2\Delta e_\mu\cdot x}{1-c^2e_\mu^2x^2}\right)\frac{\sqrt{c}}{c^\mu\sqrt{c'}}\frac{(-1)^{\mu+1}}{e^2e_2^2\cdots e_\mu}$.

If n is even, noted 2μ, we have $\theta^\mu x = \frac{x\Delta e_\mu + e_\mu\Delta x}{1-c^2e_\mu^2x^2} = \frac{x\Delta e_\mu - e_\mu\Delta x}{1-c^2e_\mu^2x^2}$, which imposes $e_\mu = 0$ or $\frac{1}{0}$. In the last case, $\theta^\mu x = \pm\frac{1}{cx}$ and $\theta^{\mu+m}x = \pm\frac{1}{c\theta^m x}$. Thus the roots of $y = \psi x$ are $x, \pm\frac{1}{cx}, \theta x, \dots, \theta^{\mu-1}x, \theta^{\mu+1}x, \dots, \theta^{2\mu-1}x$ and we have

$$p - qy = (a-by)(z-x)\left(z \mp \frac{1}{cx}\right)(z-\theta x)(z-\theta^{2\mu-1}x)\cdots$$
$$\times(z-\theta^{\mu-1}x)(z-\theta^{\mu+1}x). \tag{127}$$

We deduce from this equation that

$$a' - b'y = (by-a)\left(x \pm \frac{1}{cx} + \frac{2\Delta e\cdot x}{1-c^2e^2x^2} + \frac{2\Delta e_2\cdot x}{1-c^2e_2^2x^2} + \dots + \frac{2\Delta e_{\mu-1}\cdot x}{1-c^2e_{\mu-1}^2x^2}\right)$$

where a' and b' are the coefficients of $z^{2\mu-1}$ in p and q. It results for y a rational expression in x, invariant by $x \mapsto \theta x$. Choosing $a = b' = 0$, we obtain

$$y = \frac{a'}{b}\frac{1}{x \pm \frac{1}{cx} + \frac{2\Delta ex}{1-c^2e^2x^2} + \dots + \frac{2\Delta e_{\mu-1}x}{1-c^2e_{\mu-1}^2x^2}}$$
$$= A\frac{x(1-c^2e^2x^2)(1-c^2e_2^2x^2)\cdots(1-c^2e_{\mu-1}^2x^2)}{1+a_1x^2+a_2x^2+\dots+a_\mu x^{2\mu}} = A\varphi x.$$

If, for instance, $y = 1$ when $x = 1$, we have $A = \frac{1}{\varphi(1)}$ and, from (125), $q - p = (1-z)(1 \mp cz)\rho^2$ where ρ is a polynomial in z. Since q is even and p odd, $q + p = (1+z)(1 \pm cz)\rho'^2$ and

$$q^2 - p^2 = (1-z^2)(1-c^2z^2)(\rho\rho')^2.$$

It results that $q^2 - c'^2p^2$ must be a square and $c' = \frac{1}{\alpha}$, where α is the value of y corresponding to $x = \frac{1}{\sqrt{\pm c}}$, satisfies to this condition. Indeed $\theta^{\mu+m}x = \theta\left(\pm\frac{1}{cx}\right) = \theta\left(\frac{1}{\sqrt{\pm c}}\right) = \theta x$ for $x = \frac{1}{\sqrt{\pm c}}$, so that $p - \alpha q$ is a square and the same may be said of $p+\alpha q$. Thus $p^2 - \alpha^2 q^2 = t^2$ where t is a polynomial in z and $(q^2-p^2)(q^2-c'^2p^2) = (1-z^2)(1-c^2z^2)r^2$ for $c' = \frac{1}{\alpha}$. Sylow observes that α is never 0 nor ∞, but it is equal to 1 when μ is even and this value does not work for c'. He explains how to find a correct value in this case (*Œuvres*, t. II, p. 520–521). Then $\frac{dy}{\Delta' y} = \varepsilon\frac{dx}{\Delta x}$ where ε is the value of $\frac{dy}{dx}$ for $x = 0$, that is $\varepsilon = A = \frac{1}{\varphi(1)}$. Abel gives an expression of the denominator q of φx as a product $b(z-\delta)(z-\theta\delta)\cdots(z-\theta^{2\mu-1}\delta)$ where δ is a pole of y. It is easy to see that $\delta = \frac{1}{\sqrt{\mp c}}$ is such a pole. Thus, if e is a pole of e_μ such the equations $e_m = 0$ and $\Delta e_m = 1$ cannot be satisfied for any divisor m of 2μ, the formulae

$$\pm\frac{\varepsilon}{c}\frac{1}{y} = x \pm \frac{1}{cx} + \frac{2\Delta e x}{1-c^2e^2x^2} + \frac{2\Delta e_2 x}{1-c^2e_2^2x^2} + \ldots + \frac{2\Delta e_{\mu-1}x}{1-c^2e_{\mu-1}^2x^2},$$

$$\pm\varepsilon = c\left(1 \pm \frac{1}{c} + \frac{2\Delta e}{1-c^2e^2} + \frac{2\Delta e_2}{1-c^2e_2^2} + \ldots + \frac{2\Delta e_{\mu-1}}{1-c^2e_{\mu-1}^2}\right)$$

lead to $\frac{dy}{\sqrt{(1-y^2)(1-c'2y^2)}} = \frac{\varepsilon dx}{\sqrt{(1-x^2)(1-c^2x^2)}}$. For instance, when $\mu = 1$, $\varepsilon = 1 \pm c$, $y = (1 \pm c)\frac{x}{1 \pm cx^2}$ and $c' = \frac{2\sqrt{\pm c}}{1 \pm c}$.

Another possible value for e is a root of $e_\mu = 0$ such that $\Delta e_\mu = -1$ (for $\Delta e_\mu = 1$ would lead to $\theta^\mu x = x$). Here $\theta^\mu x = -x$, $\theta^{\mu+m}x = -\theta^m x$ and equation (127) is replaced by

$$p - qy = (a-by)(z^2-x^2)(z^2-(\theta x)^2)\cdots(z^2-(\theta^{\mu-1}x)^2)$$

which gives $a' - b'y = \pm(a-by)(x\theta x\cdots\theta^{\mu-1}x)^2$ for $z = 0$, a' and b' denoting the constant terms of p and q. Thus y is a rational function of degree 2μ of x and it remains to determine a, b, a', b' and c', ε. For instance, when $\mu = 1$, Abel finds $y = \frac{1+cx^2}{1-cx^2}$, $c' = \frac{1-c}{1+c}$, $\varepsilon = (1+c)\sqrt{-1}$ and he also gives the 5 other possible values for c'.

When the equation $y = \psi x$ has other roots than $x, \theta x, \ldots, \theta^{n-1}x$, Abel shows that the degree μ of this equation is a multiple mn of n and that its roots may be distributed in m cycles $x^{(j)}, \theta x^{(j)}, \ldots, \theta^{n-1}x^{(j)}, 0 \le j \le m-1$. The proof is identical with that used for the second theorem of the *Mémoire sur une classe particulière d'éqations ...* published in the same volume of *Crelle's Journal*. According to the precedind results, there exists a rational function $y_1 = \psi_1 x$ such that the roots of the equation $y_1 = \psi_1 x$ are $x, \theta x, \dots, \theta^{n-1}x$ and that, for convenient c_1, ε_1

$$\frac{dy_1}{\sqrt{(1-y_1^2)(1-c_1^2y_1^2)}} = \varepsilon_1 \frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}. \tag{128}$$

Let $\psi_1 z = \frac{p'}{q'}$, so that $p' - q'y = (a' - b'y)(z-x)(z-\theta x)\cdots(z-\theta^{n-1}x)$. If $y_{j+1} = \psi_1 x^{(j)}$ $(0 \le j \le m-1)$, we see that $\frac{p-qy}{a-by} = \frac{p'-q'y_1}{a'-b'y_1}\frac{p'-q'y_2}{a'-b'y_2}\cdots\frac{p'-q'y_m}{a'-b'y_m}$. Now let α be a zero and β a pole of ψz and let $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_m$ be the corresponding values of $y_1, y_2, \dots, y_m$; from the preceding relation we deduce that

$$p = A'(p' - \alpha_1 q')(p' - \alpha_2 q')\cdots(p' - \alpha_m q') \quad \text{and}$$
$$q = A''(p' - \beta_1 q')(p' - \beta_2 q')\cdots(p' - \beta_m q')$$

where A' and A'' are constants, and this gives $y = A\frac{(y_1-\alpha_1)(y_1-\alpha_2)\cdots(y_1-\alpha_m)}{(y_1-\beta_1)(y_1-\beta_2)\cdots(y_1-\beta_m)}$, rational function of degree m of y_1 where $A = \frac{A'}{A''}$. The combination of (124) and (128) gives the equation

$$\frac{dy}{\sqrt{(1-y^2)(1-c'^2y^2)}} = \frac{\varepsilon}{\varepsilon_1}\frac{dy_1}{\sqrt{(1-y_1^2)(1-c_1^2y_1^2)}}$$

and we see that the transformation of order $\mu = mn$ is obtained by composing a transformation ψ_1 of degree n and a transformation of order m. This result permits to reduce the theory of transformations to the case in which the order is a prime number.

In the general case, by the above reasoning $y = A\frac{(x-\alpha)(x-\alpha')\cdots(x-\alpha^{(\mu-1)})}{(x-\beta)(x-\beta')\cdots(x-\beta^{(\mu-1)})}$ where $\alpha, \alpha', \dots, \alpha^{(\mu-1)}$ are the zeros and $\beta, \beta', \dots, \beta^{(\mu-1)}$ the poles of ψx. Abel considers in particular the cases in which b or a is 0. When $b = 0$, the equation

$$p - qy = a(z-x)(z-x')\cdots(z-x^{(\mu-1)}) \tag{129}$$

implies that $a' - b'y = -a(x + x' + \dots + x^{(\mu-1)})$ where a' and b' are the respective coefficients of $z^{\mu-1}$ in p and q. If $\frac{x\Delta e_m + e_m \Delta x}{1-c^2e_m^2x^2} \neq \frac{x\Delta e_m - e_m\Delta x}{1-c^2e_m^2x^2}$ for all m, $\mu = 2n+1$ is odd, $a' = 0$ and

$$y = Ax\left(1 + \frac{2\Delta e_1}{1-c^2e_1^2x^2} + \dots + \frac{2\Delta e_n}{1-c^2e_n^2x^2}\right).$$

Therefore $q = (1-c^2e_1^2x^2)\cdots(1-c^2e_n^2x^2)$ and p is obtained by making $x = 0$ in (129):

$$p = az(z^2 - e_1^2)\cdots(z^2-e_n^2) \quad \text{and}$$
$$y = a\frac{x(e_1^2-x^2)(e_2^2-x^2)\cdots(e_n^2-x^2)}{(1-c^2e_1^2x^2)(1-c^2e_2^2x^2)\cdots(1-c^2e_n^2x^2)}.$$

On the contrary, if $\frac{x\Delta e+e\Delta x}{1-c^2e^2x^2} = \frac{x\Delta e-e\Delta x}{1-c^2e^2x^2}$, $e = 0$ or $\frac{1}{0}$. When $e = \frac{1}{0}$, $x' = \pm\frac{1}{cx}$, $\mu = 2n$ is even, $a' = 0$ and $y = A\left(x \pm \frac{1}{cx} + \frac{2x\Delta e_1}{1-c^2e_1^2x^2} + \ldots + \frac{2x\Delta e_{n-1}}{1-c^2e_{n-1}^2x^2}\right) = \frac{a(1-\delta_1^2x^2)(1-\delta_2^2x^2)\cdots(1-\delta_n^2x^2)}{x(1-c^2e_1^2x^2)(1-c^2e_2^2x^2)\cdots(1-c^2e_{n-1}^2x^2)}$. When $e = 0$, $x' = -x$ and one finds that p and q have the same degree, contrary to the hypothesis.

When $a = 0$, $p - qy = by(z - x)(z - x')\cdots(z - x^{(\mu-1)})$ and it results that

$$y = a\frac{(1 - c^2e_1^2x^2)(1 - c^2e_2^2x^2)\cdots(1 - c^2e_n^2x^2)}{x(e_1^2 - x^2)(e_2^2 - x^2)\cdots(e_n^2 - x^2)} \quad \text{or}$$
$$a\frac{x(1 - c^2e_1^2x^2)(1 - c^2e_2^2x^2)\cdots(1 - c^2e_{n-1}^2x^2)}{(1 - \delta_1^2x^2)(1 - \delta_2^2x^2)\cdots(1 - \delta_n^2x^2)}$$

according to the parity of μ.

In particular

$$x_{2\mu+1} = a\frac{x(e_1^2 - x^2)(e_2^2 - x^2)\cdots(e_n^2 - x^2)}{(1 - c^2e_1^2x^2)(1 - c^2e_2^2x^2)\cdots(1 - c^2e_n^2x^2)}$$
$$= A\left(x + \frac{2\Delta e_1 x}{1 - c^2e_1^2x^2} + \frac{2\Delta e_2 x}{1 - c^2e_2^2x^2} + \ldots + \frac{2\Delta e_n x}{1 - c^2e_n^2x^2}\right)$$

where $2n = (2\mu + 1)^2 - 1$. Doing $x = \frac{1}{0}$ and 0, one finds $Ac^{2n}e_1^2e_2^2\cdots e_n^2 = a$, $A = \frac{1}{2\mu+1}$ and $ae_1^2e_2^2\cdots e_n^2 = 2\mu+1$. Thus $e_1^2e_2^2\cdots e_n^2 = \frac{2\mu+1}{c^n}$ and $a = c^n = c^{2\mu^2+2\mu}$. The roots of the equation $x_{2\mu+1} = y$ are x, $\frac{x\Delta e_1\pm e_1\Delta x}{1-c^2e_1^2x^2}$, $\frac{x\Delta e_2\pm e_2\Delta x}{1-c^2e_2^2x^2}$, ..., $\frac{x\Delta e_n\pm e_n\Delta x}{1-c^2e_n^2x^2}$. Let $\theta x = \frac{x\Delta e+e\Delta x}{1-c^2e^2x^2}$ and $\theta_1 x = \frac{x\Delta e'+e'\Delta x}{1-c^2e'^2x^2}$ be two of these roots such that neither e nor e' is a root of $x_{2m+1} = 0$ for a divisor $2m + 1$ of $2\mu + 1$ and such that $\theta_1 x$ is different from $x, \theta x, \ldots, \theta^{2\mu}x$. Then $x, \theta x, \ldots, \theta^{2\mu}x, \theta_1 x, \ldots, \theta_1^{2\mu}x$ are $4\mu + 1$ distinct roots of $x_{2\mu+1} = \psi x = y$. Thus, for any m and k, $\psi(\theta^m x) = \psi(\theta_1^k x)$ and it results that $\psi(\theta_1^k\theta^m x) = \psi(\theta^{2m}x) = x_{2\mu+1}$, so that $\theta_1^k\theta^m x$ is also a root. Now it is easy to prove that, for $0 \leq m, k \leq 2\mu$, all these roots are different when $2\mu + 1$ is a prime number. We have thus written the $(2\mu + 1)^2$ roots of our equation. Their expression is

$$\theta_1^k\theta^m x = \frac{x\Delta e_{m,k} + e_{m,k}\Delta x}{1 - c^2e_{m,k}^2x^2} \quad \text{where } e_{m,k} = \frac{e_m\Delta e'_k + e'_k\Delta e_m}{1 - c^2e_m^2e_k'^2}.$$

The roots of the equation $x_{2\mu+1} = 0$ are the $e_{m,k}$, where $e_{0,0} = 0$. The non-zero roots are given by an equation of degree $4\mu^2 + 4\mu$ which may be decomposed in $2\mu + 2$ equations of degree 2μ with the help equations of degree $2\mu + 2$. It is the result of the *Recherches* of 1827 (see our §3), demonstrated here by a purely algebraic way. Indeed, if p is a rational symmetric function of $e_1, e_2, \ldots, e_{2\mu}$, it may be expressed as a rational function φe_1 of e_1 such that $\varphi e_1 = \varphi e_2 = \ldots = \varphi e_{2\mu}$. Replacing e_1 by $e_{m,1}$, we see that $\varphi e_{m,1} = \varphi e_{m,2} = \ldots = \varphi e_{2\mu m,2\mu}$. It results that the sums $\rho_k = (\varphi e_1)^k + (\varphi e_{0,1})^k + \ldots + (\varphi e_{2\mu,1})^k$ are rational symmetric in the $4\mu^2 + 4\mu$

quantities $e_{m,k}$ different from 0 and so rational functions of c. Thus p is the root of an algebraic equation of degree $2\mu + 2$ with coefficients rational in c. We may apply this result to the coefficients of the algebraic equation of which the roots are $e_1, e_2, \ldots, e_{2\mu}$.

According to the formula (126), the modulus c' obtained from c by a transformation of order $2\mu + 1$ is a rational symmetric function of $e_1, e_2, \ldots, e_{2\mu}$. It is thus a root of an equation of degree $2\mu + 2$ (the modular equation). Abel once more says that this equation seems not to be solvable by radicals. He adds that, since $\frac{dx_{2\mu+1}}{\Delta x_{2\mu+1}} = \frac{2\mu+1}{\varepsilon} \frac{dy}{\Delta' y}$, the multiplication by $2\mu + 1$ (which is of degree $(2\mu + 1)^2$) may be decomposed in the transformation of order $2\mu + 1$ from x to y and another transformation of the same order from y to $x_{2\mu+1}$. Jacobi also used such a decomposition. The expressions of $x_{2\mu+1}$ and c in y and c' are given by (126) with a root e' determined from c' as e was from c. Thus the modular equation is symmetric in (c, c').

Abel recalls the total number of transformed moduli for a given order μ: 6 for $\mu = 1$, 18 for $\mu = 2$ and $6(\mu + 1)$ for μ an odd prime number. Then he explains the algebraic solution of the equation $y = \psi x$ where ψx is a rational function defining a transformation. It is sufficient to consider the case in which the order is an odd prime number $2\mu + 1$ and we know that, in this case, the roots are $x, \theta x, \ldots, \theta^{2\mu} x$ where $\theta^m x = \frac{x \Delta e_m + e_m \Delta x}{1 - c^2 e_m^2 x^2}$ and $\theta^{2\mu+1} x = x$. Let δ be a root of 1 and $v = x + \delta\theta x + \delta^2\theta^2 x + \ldots + \delta^{2\mu}\theta^{2\mu} x$, $v' = x + \delta\theta^{2\mu} x + \delta^2\theta^{2\mu-1} x + \ldots + \delta^{2\mu}\theta x$. They are of the form $v = p + q\Delta x$, $v' = p - q\Delta x$ where p and q are rational functions of x and $vv' = s$, $v^{2\mu+1} + v'^{2\mu+1} = t$ are rational functions of x. Since they are invariant by $x \to \theta x$, they are rational functions of y and we have $v = \sqrt[2\mu+1]{\frac{t}{2} + \sqrt{\frac{t^4}{4} - s^{2\mu+1}}}$. If $v_0, v_1, \ldots, v_{2\mu}$ are the values of v corresponding to the $2\mu + 1$ roots of 1, we obtain $x = \frac{1}{2\mu+1}(v_0 + v_1 + \ldots + v_{2\mu})$, $\theta^m x = \frac{1}{2\mu+1}(v_0 + \delta^{-m} v_1 + \ldots + \delta^{-1m\mu} v_{2\mu})$.

The last chapter of this first part deals with the following problem: "Given an elliptic integral of arbitrary modulus, to express this function by means of other elliptic integrals in the most general way." According to the results of the second chapter, this problem is expressed by the equation $\int \frac{rdx}{\Delta x} = k_1\psi_1 y_1 + k_2\psi_2 y_2 + \ldots + k_m\psi_m y_m + V$ where $\varphi x = \int \frac{rdx}{\Delta x}$ is the given integral, $\psi_1, \psi_2, \ldots, \psi_m$ are elliptic integrals of respective moduli $c_1, c_2, \ldots, c_m$, $y_1, y_2, \ldots, y_m$, $\frac{\Delta_1 y_1}{\Delta x}, \frac{\Delta_2 y_2}{\Delta x}, \ldots, \frac{\Delta_m y_m}{\Delta x}$ are rational functions of x and V is an algebraic and logarithmic function. One may suppose that the number m is minimal and, according to a theorem of the fourth chapter, one has $\frac{dy_1}{\Delta_1 y_1} = \varepsilon_1 \frac{dx}{\Delta x}$, $\frac{dy_2}{\Delta_2 y_2} = \varepsilon_2 \frac{dx}{\Delta x}, \ldots, \frac{dy_m}{\Delta_m y_m} = \varepsilon_m \frac{dx}{\Delta x}$ where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ are constant. Now, for $1 \leq j \leq m$, there exists a rational function x_j of x such that $\varpi(x_j, c) = \varepsilon^{(j)}\varpi(x, c_j)$ and it results that there exits a rational function y of x such that φy be expressed as an elliptic integral of modulus c_j where x is the variable.

The part of the memoir published in Crelle's Journal stops here and Sylow completed it with a manuscript written by Abel and discovered in 1874. Here the transformation of elliptic integrals of the second and third kinds is explained. For the second kind, Abel proposes two methods. The first one is based on the differentiation with respect to the modulus c of the equation $\varpi(y, c') = \varepsilon\varpi(x, c)$, which gives

$$c'\frac{dc'}{dc}\int\frac{y^2dy}{(1-c'^2y^2)\Delta(y,c')}+\frac{dy}{dc}\frac{1}{\Delta(y,c')}$$
$$=\frac{d\varepsilon}{dc}\int\frac{dx}{\Delta(x,c)}+c\varepsilon\int\frac{x^2dx}{(1-c^2x^2)\Delta(x,c)}\,. \qquad (130)$$

Now one can verify that $\int\frac{x^2dx}{(1-c^2x^2)\Delta(x,c)}=\frac{1}{c^2-1}\frac{x(1-x^2)}{\Delta(x,c)}+\frac{1}{1-c^2}\int\frac{(1-x^2)dx}{\Delta(x,c)}$ and there is a similar identity for $\int\frac{y^2dy}{(1-c'^2y^2)\Delta(y,c')}$. Thus (130) is rewritten

$$\frac{c'}{1-c'^2}\frac{dc'}{dc}\left(\varpi(y,c')-\varpi_0(y,c')-\frac{y(1-y^2)}{\Delta(y,c')}\right)+\frac{dy}{dc}\frac{1}{\Delta(y,c')}$$
$$=\frac{d\varepsilon}{dc}\varpi(x,c)+\frac{c\varepsilon}{1-c^2}\left(\varpi(x,c)-\varpi_0(x,c)-\frac{x(1-x^2)}{\Delta(x,c)}\right)$$

or $\varpi_0(y,c')=A\varpi(x,c)+B\varpi_0(x,c)+p$ where $A=\varepsilon\left(1-\frac{cdc(1-c'^2)}{c'dc'(1-c^2)}\right)-\frac{d\varepsilon(1-c'^2)}{c'dc'}$, $B=\frac{\varepsilon c(1-c'^2)dc}{c'(1-c^2)dc'}$ and $p=\frac{(1-c'^2)dc}{c'dc'}\frac{dy}{dc}\frac{1}{\Delta(y,c')}+B\frac{x(1-x^2)}{\Delta(x,c)}-\frac{y(1-y^2)}{\Delta(y,c')}$.

The second method is based on the decomposition of y^2 in partial fractions:

$$y^2=\frac{A}{(x-a)^2}+\frac{B}{x-a}+S$$

where a is a pole of y and A, B are constant. If $y=\frac{1}{\varphi x}$, $A=\frac{1}{(\varphi' a)^2}$ and $B=-\frac{\varphi'' a}{(\varphi' a)^3}$ and we have

$$(1-x^2)(1-c^2x^2)(\varphi' x)^2=\varepsilon^2\left((\varphi x)^2-1\right)\left((\varphi x)^2-c'^2\right). \qquad (131)$$

For $x=a$, this gives $(1-a^2)(1-c^2a^2)(\varphi' a)^2=\varepsilon^2c'^2$. Let us differentiate (131) and make $x=a$; we obtain $2(1-a^2)(1-c^2a^2)\varphi' a\varphi'' a-\left(2(1+c^2)a-4c^2a^3\right)(\varphi' a)^2=0$ and we conclude that

$$A=\frac{1}{(\varphi' a)^2}=\frac{(1-a^2)(1-c^2a^2)}{\varepsilon^2c'^2},\quad B=-\frac{\varphi'' a}{(\varphi' a)^3}=\frac{-(1+c^2)a+2c^2a^3}{\varepsilon^2c'^2}.$$

Thus

$$\int\frac{y^2dy}{\Delta(y,c')}=\frac{1}{\varepsilon c'^2}\int\left(\frac{(1-a^2)(1-c^2a^2)}{(x-a)^2}+\frac{2c^2a^3-(1+c^2)a}{x-a}\right)\frac{dx}{\Delta(x,c)}$$
$$+\varepsilon\int\frac{Sdx}{\Delta(x,c)}\,. \qquad (132)$$

Now $d\frac{\Delta(x,c)}{x-a}=-\left(\frac{(1-a^2)(1-c^2a^2)}{(x-a)^2}+\frac{2c^2a^3-(1+c^2)a}{x-a}+c^2a^2-c^2x^2\right)\frac{dx}{\Delta(x,c)}$ and (132) takes the form:

$$\int \frac{y^2dy}{\Delta(y,c')} = \frac{1}{\varepsilon c'^2}\left(\frac{\Delta(x,c)}{a-x} - c^2a^2\varpi(x,c) + c^2\varpi_0(x,c)\right) + \varepsilon\int\frac{Sdx}{\Delta(x,c)} .$$

If the poles of y are $a_1, a_2, \ldots, a_\mu$, we finally obtain

$$\varepsilon c'^2\varpi_0(y,c') = \mu\varpi_0(x,c) - (c^2(a_1^2+a_2^2+\ldots+a_\mu^2) - \varepsilon^2c'^2k^2)\varpi(x,c)$$
$$+\Delta(x,c)\left(\frac{1}{a_1-x}+\frac{1}{a_2-x}+\ldots+\frac{1}{a_\mu-x}\right)$$

where k is the value of y for x infinite. Abel separately considers the cases in which $k=0$ or $k=\frac{1}{0}$. This last case is reduced to the first one by putting $x=\frac{1}{cz}$. For example, when

$$c' = \frac{2\sqrt{c}}{1+c},\ y=(1+c)\frac{x}{1+cx^2} \text{ and } \varepsilon = 1+c ,$$

$\varpi_0(y,c') = \frac{c(1+c)}{2}\varpi_0(x,c) + \frac{1+c}{2}\varpi(x,c) - \frac{1+c}{2}\frac{x\Delta(x,c)}{1+cx^2}$.

For the integral of the third kind, Abel uses the equation

$$\int\frac{dy}{(a'-x)\Delta(y,c')} = \frac{1}{a'}\Pi(y,c',a') + \int\frac{ydy}{(a'^2-y^2)\Delta(y,c')}$$

and the decomposition in partial fractions

$$\frac{1}{a'-y} = k' + \frac{1}{\varepsilon\Delta(a',c')}\left(\frac{\Delta(a_1,c)}{a_1-x}+\frac{\Delta(a_2,c)}{a_2-x}+\ldots+\frac{\Delta(a_\mu,c)}{a_\mu-x}\right)$$

which lead to

$$\frac{\Delta(a',c')}{a'}\Pi(y,c',a') + \Delta(a',c')\int\frac{ydy}{(a'^2-y^2)\Delta(y,c')}$$
$$= k_1\varpi(x,c) + \sum\frac{\Delta(a,c)}{a}\Pi(x,c,a) + v$$

where k_1 is a constant and v is an algebraic and logarithmic function. Now the sum of μ integrals in the right hand side may be reduced to a single integral with the help of the result of the third chapter: if α is determined by

$$(fx)^2 - (\varphi x)^2(\Delta(x,c))^2 = (x^2-a_1^2)(x^2-a_2^2)\cdots(x^2-a_\mu^2)(x^2-\alpha^2)$$

where fx and φx are polynomials, one even and the other odd, according to (121) we have $\sum\frac{\Delta(a,c)}{a}\Pi(x,c,a) = k_2\varpi(x,c) + \frac{\Delta(\alpha,c)}{\alpha}\Pi(x,c,\alpha) - \frac{1}{2}\log\frac{fx+\varphi x\cdot\Delta(x,c)}{fx-\varphi x\cdot\Delta(x,c)}$. The coefficients of fx and φx are determined by the equations $fa_m + \varphi a_m\cdot\Delta(a_m,c) = 0 (1\le m\le\mu)$ and the sign of $\Delta(\alpha,c)$ by $f\alpha+\varphi\alpha\Delta(\alpha,c)=0$. Another way to do this reduction consists in observing that if a is any one of $a_1, a_2, \ldots, a_\mu$, that is a root of $a'=\psi(x)$, any other has the form $a_m = \frac{a\Delta(e_m,c)+e_m\Delta(a,c)}{1-c^2e_m^2a^2}$ where e_m does not depend of a. The same formula (121) with $n=3$ and $m_1=m_2=m_3=1$ gives

$$\frac{\Delta(a_m, c)}{a_m}\Pi(x, c, a_m) = \frac{\Delta(a, c)}{a}\Pi(x, c, a) + \beta_m \varpi(x, c) + \frac{\Delta(e_m, c)}{e_m}\Pi(x, c, e_m) + \log S_m$$

and Abel shows that $\sum \frac{\Delta(e_m,c)}{e_m}\Pi(x, c, e) = 0$.

A posthumous paper, *Mémoire sur les fonctions transcendantes de la forme* $\int ydx$, *où y est une fonctions algébrique de x* (*Œuvres*, t. II, p. 206−216) contains extensions of the preceding results to more general Abelian integrals. Abel first considers μ integrals $r_j = \int y_j dx (1 \leq j \leq \mu)$ where y_j is an algebraic function of x and he supposes that they are related by an algebraic relation $R = \varphi(r_1, r_2, \ldots, r_\mu) = 0$ where φ is a polynomial with coefficients algebraic with respect to x and μ is minimal. He proves that in that case there is a *linear* relation

$$c_1 r_1 + c_2 r_2 + \ldots + c_\mu r_\mu = P \tag{133}$$

where $c_1, c_2, \ldots, c_\mu$ are constant and P is a rational function of $x, y_1, y_2, \ldots, y_\mu$. Indeed, one may suppose that $R = r_\mu^k + Pr_\mu^{k-1} + P_1 r_\mu^{k-2} + \ldots$ is irreducible with respect to r_μ (the coefficients $P, P_1, \ldots$ being rational with respect to $r_1, r_2, \ldots, r_{\mu-1}$). By differentiation, one obtains

$$r_\mu^{k-1}(ky_\mu + P') + \big((k-1)Py_\mu + P_1'\big)\, r_\mu^{k-2} + \ldots = 0\ ,$$

hence $ky_\mu + P' = 0$ and $kr_\mu + P =$ constant. This gives $k = 1$ and $R = r_\mu + P = 0$. Now the decomposition of P in partial fractions with respect to $r_{\mu-1}$ has the form

$$P = \sum \frac{S_k}{(r_{\mu-1} + t_k)^k} + \sum v_k r_{\mu-1}^k,$$

where t_k and v_k are rational with respect to $r_1, r_2, \ldots, r_{\mu-2}$; by differentiation,

$$\sum \left(-\frac{kS_k(y_{\mu-1} + t_k')}{(r_{\mu-1} + t_k)^{k+1}} + \frac{S_k'}{(r_{\mu-1} + t_k)^k}\right) + \sum (v_k' r_{\mu-1}^k + kv_k r_{\mu-1}^{k-1} y_{\mu-1}) = -y_\mu$$

and this relation implies that $S_k = 0$ and $v_k' = 0$. Moreover, if k is not equal to 1, we must have $kv_k y_{\mu-1} + v_{k-1}' = 0$, but this would imply $kv_k r_{\mu-1} + v_{k-1} =$ constant, which is impossible. So $k = 1$ and $P = v_1 r_{\mu-1} + P_1$ where v_1 is a constant and P_1 is rational with respect to $r_1, r_2, \ldots, r_{\mu-2}$. In the same way, we obtain, with a slight change of notation, $P_j = v_{\mu-1-j} r_{\mu-1-j} + P_{j+1} (0 \leq j \leq \mu - 2)$ where $v_1, v_2, \ldots, v_{\mu-1}$ are constant and P_j is rational with respect to $r_1, r_2, \ldots, r_{\mu-1-j}$. Finally, we have $r_\mu + v_{\mu-1} r_{\mu-1} + v_{\mu-2} r_{\mu-2} + \ldots + v_1 r_1 + v_0 = 0$ where v_0 is an algebraic function of x and this gives a relation of the form (133) where P is algebraic in x. Let $P^k + R_1 P^{k-1} + \ldots = 0$ be the minimal equation of P with coefficients rational in $x, y_1, y_2, \ldots, y_\mu$. Differentiating, we get $(kdP + dR_1) P^{k-1} + ((k-1) R_1 dP + dR_2)\, P^{k-2} + \ldots = 0$ with $\frac{dP}{dx} = c_1 y_1 + c_2 y_2 + \ldots$, so that

$kdP + dR_1 = 0$ and $P = -\frac{R_1}{k} +$ constant. This gives $k = 1$ and $P = -R_1$, rational with respect to $x, y_1, y_2, \ldots, y_\mu$.

In his next theorem, Abel considers a relation

$$c_1 r_1 + c_2 r_2 + \ldots + c_\mu r_\mu = P + a_1 \log v_1 + a_2 \log v_2 + \ldots + a_m \log v_m \tag{134}$$

where $v_1, v_2, \ldots, v_m$ are algebraic functions of x and P is a rational function of $x, y_1, y_2, \ldots, y_\mu, v_1, v_2, \ldots, v_m$. If v_m is root of an equation of degree n with coefficients rational in $x, y_1, y_2, \ldots, y_\mu, v_1, v_2, \ldots, v_{m-1}$, let $v'_m, v''_m, \ldots, v_m^{(n)}$ be its n values. One has

$$\begin{aligned} c_1 r_1 + c_2 r_2 + \ldots + c_\mu r_\mu = {} & \frac{1}{n}(P' + P'' + \ldots + P^{(n)}) \\ & + a_1 \log v_1 + a_2 \log v_2 + \ldots + a_{m-1} \log v_{m-1} \\ & + \frac{1}{n} a_m \log(v'_m v''_m \cdots v_m^{(n)}) \end{aligned}$$

where $P' + P'' + \ldots + P^{(n)}$ and $v'_m v''_m \cdots v_m^{(n)}$ are rational in $x, y_1, y_2, \ldots, y_\mu, v_1, v_2, \ldots, v_{m-1}$. Iterating we finally obtain $c_1 r_1 + c_2 r_2 + \ldots + c_\mu r_\mu = P + \alpha_1 \log t_1 + \alpha_2 \log t_2 + \ldots + \alpha_m \log t_m$ where $P, t_1, \ldots, t_m$ are rational functions of $x, y_1, y_2, \ldots, y_\mu$.

In particular, if y is an algebraic function of x and $\psi(x, y)$ a rational function such that the integral $\int \psi(x, y)dx$ is algebraic in $x, y, \log v_1, \log v_2, \ldots, \log v_m$, then this integral may be expressed in the form $P + \alpha_1 \log t_1 + \alpha_2 \log t_2 + \ldots + \alpha_m \log t_m$ where $P, t_1, \ldots, t_m$ are as above. If there is a relation $\int \psi(x, y)dx + \int \psi_1(x, y_1)dx = R$ where R is of the form of the right hand side of (134), and if the minimal equation for y_1 remains irreducible after adjunction of y, then one has separately $\int \psi(x, y)dx = R_1$ and $\int \psi_{\mathrm{x}}(x, y_1)dx = R_2$. For if $y'_1, y''_1, \ldots, y_1^{(n)}$ are the values of the algebraic function x,

$$\begin{aligned} & n\psi(x, y)dx + (\psi_1(x, y'_1) + \psi_1(x, y''_1) + \ldots + \psi_1(x, y_1^{(n)}))dx \\ & = d(R' + R'' + \ldots + R^{(n)}) , \end{aligned}$$

hence a relation $\int \psi(x, y)dx = \frac{1}{n}(R' + R'' + \ldots + R^{(n)}) - \int f(x)dx = R_1$ and then $\int \psi_{\mathrm{x}}(x, y_1)dx = R - R_1 = R_2$. If there is a relation $\int ydx = R$ where $y = p_0 + p_1 s^{-\frac{1}{n}} + p_2 s^{-\frac{2}{n}} + \ldots + p_{n-1} s^{-\frac{n-1}{n}}$, $p_0, p_1, \ldots, p_{n-1}, s$ algebraic functions such that $s^{\frac{1}{n}}$ is not rational in $p_0, p_1, \ldots, p_{n-1}, s$, then one has separately $\int \frac{p_m dx}{s^{\frac{m}{n}}} = R_j$ $(0 \le m \le n-1)$. Indeed $dR = df\left(s^{\frac{1}{n}}\right) = \psi\left(s^{\frac{1}{n}}\right)dx$ and the same relation is true for any value $\alpha^k s^{\frac{1}{n}}$ of $s^{\frac{1}{n}}$ (α primitive n-th root of 1). It is easy to deduce that

$$\int \frac{p_m dx}{s^{\frac{m}{n}}} = \frac{1}{n}(f(\sqrt[n]{s}) + \alpha^m f(\alpha\sqrt[n]{s}) + \ldots + \alpha^{(n-1)m} f(\alpha^{n-1}\sqrt[n]{s})) .$$

The rest of the paper is not finished. Abel studies the cases in which an integral

$$y = \int f\,(x, (x-a_1)^{\frac{1}{m_1}}, (x-a_2)^{\frac{1}{m_2}}, \ldots, (x-a_n)^{\frac{1}{m_n}})dx$$

(f is rational) is an algebraic function, and the corresponding reductions of Abelian integrals. According to the preceding results, he is reduced to

$$\int dx \cdot p \cdot (x-a_1)^{-\frac{k_1}{m_1}}(x-a_2)^{-\frac{k_2}{m_2}}\cdots(x-a_n)^{-\frac{k_n}{m_n}}$$
$$= P = v(x-a_1)^{1-\frac{k_1}{m_1}}(x-a_2)^{1-\frac{k_2}{m_2}}\cdots(x-a_n)^{1-\frac{k_n}{m_n}}$$

where p and v are rational and $\frac{k_1}{m_1}, \frac{k_2}{m_2}, \ldots, \frac{k_n}{m_n}$ are between 0 and 1. This gives

$$p = v(A_0 + A_1x + \ldots + A_{n-1}x^{n-1}) + \frac{dv}{dx}(x-a_1)(x-a_2)\cdots(x-a_n) = v\varphi x + \frac{dv}{dx}fx$$

where

$$A_0 + A_1x + \ldots + A_{n-1}x^{n-1} = \left(1-\frac{k_1}{m_1}\right)(x-a_2)(x-a_3)\cdots(x-a_n) + \left(1-\frac{k_2}{m_2}\right)(x-a_1)(x-a_3)\cdots(x-a_n) + \ldots + \left(1-\frac{k_n}{m_n}\right)(x-a_1)(x-a_2)\cdots(x-a_{n-1}) \,.$$

Abel explains the cases in which $v = x^m$ or $\frac{1}{(x-\alpha)^m}$. In the first case

$$\begin{aligned} p &= x^m(A_0 + A_1x + \ldots + A_{n-1}x^{n-1}) \\ &\quad + mx^{m-1}(B_0 + B_1x + \ldots + B_{n-1}x^{n-1} + x^n) \\ &= mB_0x^{m-1} + (A_0 + mB_1)x^m + (A_1 + mB_2)x^{m+1} + \ldots \\ &\quad + (A_{n-1} + m)x^{n+m-1}. \end{aligned}$$

Putting $\int x^\mu dx(x-a_1)^{-\frac{k_1}{m_1}}(x-a_2)^{-\frac{k_2}{m_2}}\cdots(x-a_n)^{-\frac{k_n}{m_n}} = R_\mu$, he gets

$$R_{m+n-1} = \frac{1}{m+A_{n-1}}x^m(x-a_1)^{1-\frac{k_1}{m_1}}(x-a_2)^{1-\frac{k_2}{m_2}}\cdots(x-a_n)^{1-\frac{k_n}{m_n}} - \frac{mB_0}{m+A_{n-1}}R_{m-1} - \ldots - \frac{A_{n-2}+mB_{n-1}}{m+A_{n-1}}R_{m+n-2}$$

a recursion formula which permits to express R_{m+n-1} by $R_0, R_1, \ldots, R_{n-2}$.

In the second case

$$\begin{aligned} p &= \frac{\varphi x}{(x-\alpha)^m} - \frac{mfx}{(x-\alpha)^{m+1}} \\ &= -\frac{mf\alpha}{(x-\alpha)^{m+1}} + \frac{\varphi\alpha - mf'\alpha}{(x-\alpha)^m} + \frac{\varphi'\alpha - \frac{mf''\alpha}{2}}{(x-\alpha)^{m-1}} + \ldots \\ &\quad + \frac{\frac{\varphi^{(n-1)}\alpha}{1\cdot 2\cdots(n-1)} - m\frac{f^{(n)}\alpha}{12\cdots n}}{(x-\alpha)^{m-n+1}} \,. \end{aligned}$$

Putting $\int \frac{dx}{(x-\alpha)^\mu}(x-a_1)^{-\frac{k_1}{m_1}}(x-a_2)^{-\frac{k_2}{m_2}}\cdots(x-a_n)^{-\frac{k_n}{m_n}} = S_\mu$, he gets

$$\frac{(x-a_1)^{1-\frac{k_1}{m_1}}(x-a_2)^{1-\frac{k_2}{m_2}}\cdots(x-a_n)^{1-\frac{k_n}{m_n}}}{(x-\alpha)^m} \\ = -mf\alpha S_{m+1} + (\varphi\alpha - mf'\alpha)S_m + \ldots + \left(\frac{\varphi^{n-1}\alpha}{1\cdot 2\cdots(n-1)} - \frac{mf^{(n)}\alpha}{1\cdot 2\cdots n}\right)S_{m-n+1}.$$

If $f\alpha \neq 0$, this permits to express S_{m+1} in $S_1, R_0, R_1, \ldots, R_{n-2}$. If $f\alpha = 0$ but $\varphi\alpha - mf'\alpha \neq 0$, S_m is a linear combination of $R_0, R_1, \ldots, R_{n-2}$. Now

$$\varphi a_1 - mf'a_1 = \left(1 - \frac{k_1}{m_1} - m\right)(a_1-a_2)\cdots(a_1-a_n) \neq 0,$$

so that the S_m with 'parameter' a_1 are linear combinations of $R_0, R_1, \ldots, R_{n-2}$.

By the same method, Abel proves that a linear relation

$$c_0R_0 + c_1R_1 + \ldots + c_{n-2}R_{n-2} + \varepsilon_1 t_1 + \varepsilon_2 t_2 + \ldots + \varepsilon_\mu t_\mu \\ = v(x-a_1)^{1-\frac{k_1}{m_1}}(x-a_2)^{1-\frac{k_2}{m_2}}\cdots(x-a_n)^{1-\frac{k_n}{m_n}},$$

where $t_k = \int \frac{dx}{(x-\alpha_k)^\mu}(x-a_1)^{-\frac{k_1}{m_1}}(x-a_2)^{-\frac{k_2}{m_2}}\cdots(x-a_n)^{-\frac{k_n}{m_n}}$, is not possible. He finally proves that, in a relation $c_0R_0 + c_1R_1 + \ldots + c_{n-2}R_{n-2} + \varepsilon_1 t_1 + \varepsilon_2 t_2 + \ldots + \varepsilon_\mu t_\mu = P + \alpha_1 \log v_1 + \alpha_2 \log v_2 + \ldots + \alpha_m \log v_m$, the right hand side may be reduced to the form $\nu r_{\nu-1}\lambda_{\nu-1} + \sum \alpha \sum \omega^{k'} \log(\sum(s_k\lambda_k\omega^{k'k}))$ where ν is the g.c.d. of $m_1, m_2, \ldots, m_n$, for each $k \in [0, \nu-1]$, $\lambda_k = (x-a_1)^{\frac{\ell_1}{m_1}}(x-a_2)^{\frac{\ell_2}{m_2}}\cdots(x-a_n)^{\frac{\ell_n}{m_n}}$, λ_j being the remainder of the division of kk_j by m_j, ω is a primitive ν-th root of 1 and $r_{\nu-1}, s_0, s_1, \ldots, s_{\nu-1}$ are polynomials. First of all, the right hand side has the form

$$r_0 + r_1\lambda_1 + \ldots + r_{\nu-1}\lambda_{\nu-1} + \sum \alpha \log(s_0 + s_1\lambda_1 + \ldots + s_{\nu-1}\lambda_{\nu-1})$$

and when λ_1 is replaced by another value $\omega^{k'}\lambda_1$, λ_k becomes $\omega^{k'k}\lambda_k$. We thus get ν expressions for the considered integral $\int \frac{fx\cdot dx}{\lambda_1}$ and the terms $r_k\lambda_k$ with $k < \nu - 1$ disappear from the sum of these expressions. It is then possible to prove that $r_{\nu-1} = 0$ and that the relations of the considered type are combinations of those in which only one α is different from 0. In this case $\int \frac{fx\cdot dx}{\lambda_1} = \theta(x, \lambda_1) = \log\theta(\lambda_1) + \omega\log\theta(\omega\lambda_1) + \omega^2\log\theta(\omega^2\lambda_1) + \ldots + \omega^{\nu-1}\log\theta(\omega^{\nu-1}\lambda_1)$ where $\theta(\lambda_1) = s_0 + s_1\lambda_1 + \ldots + s_{\nu-1}\lambda_{\nu-1}$ and Abel attacks the determination of the possible forms for fx, but the paper is left incomplete (see Sylow's note, *Œuvres*, t. II, p. 327–329).

9 Series

We saw above (§3) that in his first papers, Abel did not hesitate to use infinite series in the 18th century manner, that is without any regard to questions of convergence. On the contrary, when dealing with expansions of elliptic functions (§6), he tried to treat the problem much more rigourously. In the meantime, he had read Cauchy's lectures at the *École Polytechnique* and he was impressed by this work. In a letter to Holmboe (16 January 1826), he writes "On the whole, divergent series are the work of the Devil and it is a shame that one dares base any demonstration on them. You can get whatever result you want when you use them, and they have given rise to so many disasters and so many paradoxes." Abel then explains that even the binomial formula and Taylor theorem are not well based, but that he has found a proof for the binomial formula and Cauchy's lectures contain a proof for Taylor theorem.

The memoir *Recherches sur la série* $1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \ldots$, published in the first volume of Crelle's *Journal* (1826; *Œuvres*, t. I, p. 219–250) is devoted to a rigourous and general proof of the binomial formula. We have already explained the formal part of this memoir (§1) and we shall now analyse the part where Abel studies questions of convergence. Abel defines a convergent series as a series $v_0 + v_1 + v_2 + \ldots + v_m + \ldots$ such that the partial sum $v_0 + v_1 + v_2 + \ldots + v_m$ gets indefinitely nearer to a certain limit, which is called the *sum* of the series, for increasing m, and he states Cauchy's criterium for convergence. The first theorem says that a series $\varepsilon_0\rho_0 + \varepsilon_1\rho_1 + \varepsilon_2\rho_2 + \ldots + \varepsilon_m\rho_m + \ldots$ is divergent when $\rho_0, \rho_1, \rho_2, \ldots$ are positive numbers such that $\frac{\rho_{m+1}}{\rho_m}$ has a limit $\alpha > 1$ and the ε_m do not tend towards 0. On the contrary (theorem II), if the limit α is < 1 and the ε_m remain ≤ 1, the series is convergent. The proof uses the comparison of $\rho_0 + \rho_1 + \ldots + \rho_m + \ldots$ with a convergent geometric series and Cauchy's criterium. In the third theorem, Abel considers a series

$$t_0 + t_1 + \ldots + t_m + \ldots$$

of which the partial sums $p_m = t_0 + t_1 + \ldots + t_m$ remain bounded by some quantity δ and a decreasing sequence of positive numbers $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m, \ldots$ The theorem states that

$$r = \varepsilon_0 t_0 + \varepsilon_1 t_1 + \varepsilon_2 t_2 + \ldots + \varepsilon_m t_m$$

remains bounded by $\delta\varepsilon_0$. Abel uses what is now called 'Abel transformation', putting $t_0 = p_0, t_1 = p_1 - p_0, t_2 = p_2 - p_1, \ldots$ so that $r = p_0(\varepsilon_0 - \varepsilon_1) + p_1(\varepsilon_1 - \varepsilon_2) + \ldots + p_{m-1}(\varepsilon_{m-1} - \varepsilon_m) + p_m\varepsilon_m \leq \delta\varepsilon_0$.

Theorem IV concerns a power series $f\alpha = v_0 + v_1\alpha + v_2\alpha^2 + \ldots + v_m\alpha^m + \ldots$ and it says that if the series is convergent for a (positive) value δ of α, it remains convergent for the (positive) values $\alpha \leq \delta$ and, for such an α, the limit of $f(\alpha - \beta)$ for $\beta \to 0$ is $f\alpha$. Abel puts $\varphi\alpha = v_0 + v_1\alpha + v_2\alpha^2 + \ldots + v_{m-1}\alpha^{m-1}$ and $\psi\alpha = v_m\alpha^m + v_{m+1}\alpha^{m+1} + \ldots = \left(\frac{\alpha}{\delta}\right)^m v_m\delta^m + \left(\frac{\alpha}{\delta}\right)^{m+1} v_{m+1}\delta^{m+1} + \ldots \leq \left(\frac{\alpha}{\delta}\right)^m p$ where $p \geq v_m\delta^m, v_m\delta^m + v_{m+1}\delta^{m+1}, v_m\delta^m + v_{m+1}\delta^{m+1} + v_{m+2}\delta^{m+2}, \ldots$ (theorem III), and this bound is arbitrarily small for m sufficiently large. Now $f\alpha - f(\alpha - \beta) = \varphi\alpha - \varphi(\alpha - \beta) + \psi\alpha - \psi(\alpha - \beta)$ and, since $\varphi\alpha$ is a polynomial, it is sufficient to bound $\psi\alpha - \psi(\alpha - \beta)$ by $\left(\left(\frac{\alpha}{\delta}\right)^m + \left(\frac{\alpha-\beta}{\delta}\right)^m\right) p$, which is easy to do.

In the following theorem, the coefficients $v_0, v_1, \ldots$ are continuous functions of x in an interval $[a, b]$ and Abel says that if the series is convergent for a value δ of α, its sum fx for $\alpha < \delta$ is a continuous function in $[a, b]$. Unfortunately, this theorem is not quite correct. Abel's proof consists in writing $fx = \varphi x + \psi x$ where φx is the sum of the terms up to $m-1$ and ψx is the corresponding remainder, which is bounded by $\left(\frac{\alpha}{\delta}\right)^m \theta x$ where $\theta x \geq v_m\delta^m, v_m\delta^m + v_{m+1}\delta^{m+1}, v_m\delta^m + v_{m+1}\delta^{m+1} + v_{m+2}\delta^{m+2}, \ldots$ (theorem III). For each x, this bound tends towards 0 as $m \to \infty$ but the convergence is not necessarily uniform in x and Abel's reasoning implicitly uses this uniformity. Recall that Cauchy stated more generally that the sum of a convergent series of continuous functions is continuous. In a footnote, Abel criticises this statement, giving the series $\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots$ as a counterexample: the series is everywhere convergent but its sum is discontinuous for $x = (2m+1)\pi$ (where it is 0).

Theorem VI correctly states the formula for the product of two *absolutely* convergent series $v_0 + v_1 + v_2 + \ldots = p$ and $v'_0 + v'_1 + v'_2 + \ldots = p'$. Let ρ (resp. ρ'_m) be the absolute value of v_m (resp. v'_m). The hypothesis is that $\rho_0 + \rho_1 + \rho_2 + \ldots = u$ and $\rho'_0 + \rho'_1 + \rho'_2 + \ldots = u'$ are convergent and the conclusion that the series of general term $r_m = v_0v'_m + v_1v'_{m-1} + v_2v'_{m-2} + \ldots + v_mv'_0$ is convergent and that its sum is equal to pp'. Indeed $r_0 + r_1 + r_2 + \ldots + r_{2m} = p_mp'_m + t + t'$ where

$$p_m = v_0 + v_1 + \ldots + v_m, \quad p'_m = v'_0 + v'_1 + \ldots + v'_m\ ,$$
$$t = p_0v'_{2m} + p_1v'_{2m-1} + \ldots + p_{m-1}v'_{m+1}\ ,$$
$$t' = p'_0v_{2m} + p'_1v_{2m-1} + \ldots + p'_{m-1}v_{m+1}\ .$$

Now $|t| \leq u(\rho'_{2m} + \rho'_{2m-1} + \ldots + \rho'_{m+1})$, $|t'| \leq u'(\rho_{2m} + \rho_{2m-1} + \ldots + \rho_{m+1})$ so that t and t' tend towards 0. This result had been given by Cauchy in the sixth chapter of his *Analyse algébrique* (1821).

As an application, Abel considers two convergent series $t_0 + t_1 + t_2 + \ldots$, $t'_0 + t'_1 + t'_2 + \ldots$ with real terms and such that the series $t_0t'_0 + (t_1t'_0 + t_0t'_1) + (t_2t'_0 + t_1t'_1 + t_0t'_2) + \ldots$ is also convergent. Then the sum of this last series is equal to the product of the sums of the two given series. Indeed, by theorem IV, it is the limit of $t_0t'_0 + (t_1t'_0 + t_0t'_1)\alpha + (t_2t'_0 + t_1t'_1 + t_0t'_2)\alpha^2 + \ldots$ for $\alpha \to 1$ ($\alpha < 1$). Since both series $t_0 + t_1\alpha + t_2\alpha^2 + \ldots$ and $t'_0 + t'_1\alpha + t'_2\alpha^2 + \ldots$ are absolutely convergent for $\alpha < 1$ according to theorem II, the product of their sums is equal to

$$t_0t'_0 + (t_1t'_0 + t_0t'_1)\alpha + (t_2t'_0 + t_1t'_1 + t_0t'_2)\alpha^2 + \ldots$$

and the conclusion is clear.

In the third volume of Crelle's *Journal*, Abel published a *Note sur un mémmoire de M.L. Olivier, ayant pour titre "Remarques sur les séries infinies et leur convergence"* (1828; *Œuvres*, t. I, p. 399–402). In his memoir, Olivier stated a wrong criterium for the convergence of a series $\sum a_n$: that na_n must tend towards 0. As a counterexample, Abel gives the divergent series of general term $a_n = \frac{1}{n\log n}$ for which $na_n = \frac{1}{\log n}$ tends towards 0. He proves the divergence using the inequality $\log(1 + x) < x$, which gives $\log\left(1 + \frac{1}{n}\right) < \frac{1}{n}$ or $\log\log(1 + n) < \log\log n + \log\left(1 + \frac{1}{n\log n}\right) < \log\log n + \frac{1}{n\log n}$. It results that $\log\log(1 + n) < \log\log 2 + \frac{1}{2\log 2} + \frac{1}{3\log 3} + \ldots + \frac{1}{n\log n}$ and the divergence follows from $\lim\,(\log\log(1 + n)) = \infty$.

More generally, Abel proves that there is no function φn such that $\lim(\varphi n\cdot a_n)=0$ be a criterium for the convergence of $\sum a_n$. Indeed, when $\sum a_n$ is divergent, the same is true for the series $\frac{a_1}{a_0}+\frac{a_2}{a_0+a_1}+\frac{a_3}{a_0+a_1+a_2}+\ldots+\frac{a_n}{a_0+a_1+\ldots+a_{n-1}}+\ldots$ for

$$\begin{aligned}&\log(a_0+a_1+\ldots+a_n)-\log(a_0+a_1+\ldots+a_{n-1})\\&=\log\left(1+\frac{a_n}{a_0+a_1+\ldots+a_{n-1}}\right)<\frac{a_n}{a_0+a_1+\ldots+a_{n-1}}\end{aligned}$$

and $\log(a_0+a_1+\ldots+a_n)-\log a_0<\frac{a_1}{a_0}+\frac{a_2}{a_0+a_1}+\frac{a_3}{a_0+a_1+a_2}+\ldots+\frac{a_n}{a_0+a_1+\ldots+a_{n-1}}$. Now if φn is a function such that $\varphi n\cdot a_n\to 0$ is a criterium of convergence, the series

$$\frac{1}{\varphi(1)}+\frac{1}{\varphi(2)}+\frac{1}{\varphi(3)}+\frac{1}{\varphi(4)}+\ldots+\frac{1}{\varphi n}+\ldots$$

is divergent but

$$\begin{aligned}&\frac{1}{\varphi(2).\frac{1}{\varphi(1)}}+\frac{1}{\varphi(3)\left(\frac{1}{\varphi(1)}+\frac{1}{\varphi(2)}\right)}+\frac{1}{\varphi(4)\left(\frac{1}{\varphi(1)}+\frac{1}{\varphi(2)}+\frac{1}{\varphi(3)}\right)}+\ldots\\&+\frac{1}{\varphi n\left(\frac{1}{\varphi(1)}+\frac{1}{\varphi(2)}+\frac{1}{\varphi(3)}+\ldots+\frac{1}{\varphi(n-1)}\right)}+\ldots\end{aligned}$$

is convergent, which is contradictory.

Abel left unpublished a memoir *Sur les séries* (*Œuvres*, t. II, p. 197–205), probably written at the end of 1827. He begins by giving the definition of convergence and recalling Cauchy's criterium. Then the first part deals with series of positive terms and the second part with series of functions. The first theorem states that if a series $u_0+u_1+u_2+\ldots+u_n+\ldots$ with $u_n\geq 0$ is divergent, then the same is true of $\frac{u_1}{s_0^\alpha}+\frac{u_2}{s_1^\alpha}+\frac{u_3}{s_2^\alpha}+\ldots+\frac{u_n}{s_{n-1}^\alpha}+\ldots$, where $s_n=u_0+u_1+u_2+\ldots+u_n$ and $\alpha\leq 1$. It is an immediate extension of the preceding lemma, where α was taken equal to 1. The following theorem says that, under the same hypotheses, $\sum\frac{u_n}{s_n^{1+\alpha}}$ is convergent when $\alpha>0$. Indeed $s_{n-1}^{-\alpha}-s_n^{-\alpha}=(s_n-u_n)^{-\alpha}-s_n^{-\alpha}>\alpha\frac{u_n}{s_n^{1+\alpha}}$. For example, if $u_n=1$, the first theorem gives the divergence of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$ and the second theorem gives the convergence of the series $1+\frac{1}{2^{\alpha+1}}+\frac{1}{3^{\alpha+1}}+\frac{1}{4^{\alpha+1}}+\ldots+\frac{1}{n^{\alpha+1}}+\ldots$ for $\alpha>0$. When a series $\sum\varphi n$ is divergent, a necessary condition for the convergence of $\sum u_n$ is that

$$\liminf\frac{u_n}{\varphi n}=0\,.$$

Indeed, if it is not the case, there exists $\alpha>0$ such that $p_n=\frac{u_n}{\varphi n}\geq\alpha$ for n large enough and $\sum u_n\geq\sum\alpha\cdot\varphi n$ is divergent. Thus $\sum u_n$ is convergent only if $\liminf nu_n=0$ but this condition is not sufficient and Abel recalls the final result of the preceding memoir. Abel next considers a function φn increasing without limit, implicitly supposed to be differentiable and concave, so that $\varphi(n+1)-\varphi n \leq \varphi' n$ and $\varphi'(0)+\varphi'(1)+\ldots+\varphi'(n) > \varphi(n+1)-\varphi(0)$ and this implies the divergence of $\varphi'(0)+\varphi'(1)+\ldots+\varphi'(n)+\ldots$ This applies to the iterated logarithm $\varphi_m n = \log^m(n+a)$: $\varphi_n' n = \frac{1}{(n+a)\log(n+a)\log^2(n+a)\cdots\log^{m-1}(n+a)}$ and the series

$$\sum \frac{1}{n \log n \log^2 n \cdots \log^{m-1} n}$$

is divergent. On the contrary, when $\varphi n = C - \frac{1}{\alpha-1}\frac{1}{(\log^m n)^{\alpha-1}}$ where $\alpha > 1$, $\varphi(n+1)-\varphi n > \varphi'(n+1)$ and $\varphi' n < \frac{1}{\alpha-1}\left(\frac{1}{(\log^m(n-1))^{\alpha-1}} - \frac{1}{(\log^m n)^{\alpha-1}}\right)$. It results that $\varphi'(a)+\varphi'(a+1)+\ldots+\varphi' n < \frac{1}{\alpha-1}\frac{1}{(\log^m(a-1))^{\alpha-1}}$ and the series $\sum \frac{1}{n\log n \log^2 n \cdots \log^{m-1} n (\log^m n)^{1+\alpha}}$ is convergent for $\alpha > 0$. Abel derives from this statement a rule for the convergence of a series $\sum u_n$: the series is convergent if $\lim \frac{\log\left(\frac{1}{u_n n \log n \cdots \log^{m-1} n}\right)}{\log^{m+1} n} > 1$ and it is divergent if this limit is < 1. For instance, in the first case, there exists an $\alpha > 0$ such that $u_n < \frac{1}{n \log n \cdots \log^{m-1} n (\log^m n)^{1+\alpha}}$ for n large enough.

The first result stated by Abel on the series of functions is that when a power series $\sum a_n x^n$ converges in $]-\alpha, \alpha[$, it may be differentiated term by term in this interval. Abel returns to theorem V of his memoir on the binomial formula, which shows that he was not satisfied with its proof. He considers $\varphi_0(y)+\varphi_1(y)x+\varphi_2(y)x^2+\ldots+\varphi_n(y)x^n+\ldots = f(y)$ and he supposes that it is convergent for $0 \leq x < \alpha$ and y near a value β. Let A_n be the limit of $\varphi_n(y)$ when y tends towards β and suppose that $A_0 + A_1 x + \ldots + A_n x^n + \ldots$ is convergent. Then the sum R of this series is the limit of $f(y)$. Abel writes

$$\begin{aligned} f(\beta-\omega) - R = {} & (\varphi_0(\beta-\omega) - A_0) + (x_1\varphi_1(\beta-\omega) - A_1 x_1)\, x_2 + \ldots \\ & + \left(\varphi_n(\beta-\omega)x_1^n - A_1 x_1^n\right) x_2^n + \ldots \end{aligned}$$

where $x = x_1 x_2$, $x_1 < \alpha$, $x_2 < 1$ and ω tends towards 0 and he chooses m such that

$$\varphi_m(\beta-\omega)x_1^m - A_1 x_1^m \geq \varphi_n(\beta-\omega)x_1^n - A_1 x_1^n$$

for all n, so that $f(\beta-\omega) = R + \frac{k}{1-x_2}(\varphi_m(\beta-\omega)x_1^m - A_1 x_1^m)$ where $-1 \leq k \leq 1$. Unfortunately, the value of m may depend on ω and the proof is still insufficient. As Lie remarks in the final notes (*Œuvres*, t. II, p. 326), it is sufficient to suppose that there exists M such that

$$(\varphi_n(\beta-\omega) - A_n)\,\alpha_0^n \leq M$$

for all n, for $x_1 < \alpha_0 < \alpha$ and for ω small enough in order to restaure a correct proof. Abel applies his theorem to the series

$$1^y x + 2^y x^2 + 3^y x^3 + \ldots + n^y x^n + \ldots ,$$
$$\sin y \cdot x + \frac{1}{2}\sin 2y \cdot x^2 + \frac{1}{3}\sin 3y \cdot x^3 + \ldots ,$$
$$\frac{y}{1+y^2} + \frac{y}{4+y^2}x + \frac{y}{9+y^2}x^2 + \ldots$$

continuous functions of $y \in \mathbb{R}$ when $0 \le x < 1$; the second one is still convergent when $x = 1$, but its sum has discontinuities as a function of y. The third one has 0 for limit when y tends towards ∞, if $x < 1$, but the limit is $\frac{\pi}{2}$ if $x = 1$. Abel adds two remarks:

I. the series $\frac{\sin ay}{y} + \frac{\sin a^2 y}{y}x + \ldots + \frac{\sin a^{n+1} y}{y}x^n + \ldots$ is convergent for $0 \le x < 1$ and $y > 0$, but when y tends towards 0, the limit A_n of $\frac{\sin a^{n+1} y}{y}$ is a^{n+1}, so that the series

$$A_0 + A_1 x + \ldots + A_n x^n + \ldots$$

is divergent when $ax > 1$.

II. the sum of

$$1+a+\ldots+a^y-(1+2a+\ldots+(y+1)a^y)x+(1+3a+\ldots+\frac{(y+1)(y+2)}{2}a^y)x^2-\ldots$$

is equal to $\frac{1}{1+x} + \frac{a}{(1+x)^2} + \ldots + \frac{a^y}{(1+x)^{y+1}} = fy$ for $0 \le x < 1$ and y integer. When $y \to \infty$, this sum has for limit $\frac{1}{1+x-a}$ if $a < 1+x$, but, if $a \ge 1$, the limit of $\varphi_n(y) = 1 + (n+1)a + \ldots + \binom{y+n}{n} a^y$ is infinite and for $a < 1$, it is $\frac{1}{(1-a)^{n+1}} = A_n$. The series $A_0 + A_1 x + \ldots + A_n x^n + \ldots$ does not converge when $1 - x \le a < 1$.

Abel gives an extension of his theorem IV of the memoir on the binomial series to the case in which $a_0 + a_1\alpha + a_2\alpha^2 + \ldots$ is divergent. In this case, if $a_n x^n$ is positive for n large, the limit of $a_0 + a_1 x + a_2 x^2 + \ldots$ for $x < \alpha$ tending towards α is infinite. The end of the paper contains a proof of Taylor theorem for a function $fx = a_0 + a_1 x + a_2 x^2 + \ldots$ defined by a power series convergent for $0 \le x < 1$. A lemma states that if

$$\begin{aligned} fx = {} & (a_0^{(0)} + a_1^{(0)}x + a_2^{(0)}x^2 + \ldots) + (a_0^{(1)} + a_1^{(1)}x + a_2^{(1)}x^2 + \ldots) + \ldots \\ & + (a_0^{(n)} + a_1^{(n)}x + a_2^{(n)}x^2 + \ldots) + \ldots \end{aligned}$$

is convergent for $0 \le x < 1$ and if $A_0 = a_0^{(0)} + a_0^{(1)} + \ldots + a_0^{(n)} + \ldots$, $A_1 = a_1^{(0)} + a_1^{(1)} + \ldots + a_1^{(n)} + \ldots, \ldots$ then $fx = A_0 + A_1 x + A_2 x^2 + \ldots + A_m x^m + \ldots$ whenever this series is convergent. Then Abel writes

$$\begin{aligned} f(x+\omega) &= a_0 + a_1(x+\omega) + a_2(x+\omega)^2 + \ldots \\ &= a_0 + a_1 x + a_2 x^2 + \ldots + (a_1 + 2a_2 x + \ldots)\omega + \ldots \\ &= fx + \frac{f'x}{1}\omega + \frac{f''x}{1\cdot 2}\omega^2 + \ldots \end{aligned}$$

if this series is convergent. It remains to prove the convergence under the condition $x + \omega < 1$. Abel writes $x + \omega = x_1$ and $x = x_1 x_2$, so that $x_2 < 1$ and

$$\begin{aligned} x_1^n \frac{f^n x}{1 \cdot 2 \cdots n} &= x_1^n a_n + (n+1) a_{n+1} x_1^{n+1} x_2 \\ &\quad + \frac{(n+1)(n+2)}{1 \cdot 2} a_{n+2} x_1^{n+2} x_2^2 + \ldots \\ &\leq v_n \frac{1}{(1-x_2)^{n+1}} \end{aligned}$$

where v_n is the least upper bound of $a_{n+k} x_1^{n+k}$ for $k \geq 0$. This gives

$$\omega^n \frac{f^n x}{1 \cdot 2 \cdots n} \leq v_n \left(\frac{\omega}{x_1 - x_1 x_2} \right)^n \frac{1}{1 - x_2} = \frac{v_n}{1 - x_2}$$

where v_n tends towards 0.

10 Conclusion

Two main subjects constitute the core of Abel's work: algebraic equations and elliptic functions, with an extension to the most general abelian integrals. As we saw, they are intimately connected. Within our modern terminology, these subjects may be symbolised by the terms 'Abelian group', which refers to a class of solvable equations discovered by Abel, that is equations with a commutative Galois group, and by the theorem of Abel on Abelian integrals and the term 'Abelian variety'.

The theory of algebraic equations was one of the earliest fields of activity of Abel. He proved the impossibility to solve by radicals the general quintic equation. But later on he discovered that the so called Abelian equations are algebraically solvable and he attacked the general problem to characterise solvable equations. He obtained important results on the form of the solutions of solvable equations, and this part of the theory was the point of departure of Kronecker's work in algebra. Galois attacked the same problem from a different point of view, introducing the Galois group which measures the indiscernability between the roots.

Abel studied elliptic integrals in Legendre's *Exercices de Calcul Intégral*, following Degen's advice, and he immediately found fundamental new results. At the same time, Jacobi began to investigate this subject and Abel was stimulated by the competition with Jacobi. His theory contains all Jacobi's results up to the year 1829, but also some results of his own, as the study of the equation of division of an elliptic integral or of a period of such an integral. Particularly important is his discovery of complex multiplication which became a favourite subject for Kronecker and one of the sources of class field theory.

Abel's extension of the addition theorem for elliptic integrals to the general case of Abelian integrals is rightly considered as one of the most important discoveries in the first half of 19th century. It led Jacobi to formulate the inversion problem for hyperelliptic integrals. Through the works of Riemann and Clebsch, it became the base of a new method to study the geometry of algebraic curves. Abel's method to prove this theorem contains in germ the notions of divisors and of linear families of divisors on an algebraic curve and Riemann's interpretation of Abel's result leads to the notion of Jacobian of an algebraic curve.

With Gauss, Bolzano, Cauchy and Dirichlet, Abel is one of the reformators of rigour in the first half of 19th century. Abel's transformation of series gave him a way to prove the continuity of the sum of a power series up to the end of the interval of convergence in the case in which the series converges in this point. This theorem is the base of a method of summation for divergent series.

Abel always tried to attack problems in the most general way instead of studying particular cases and particular objects. In the theory of algebraic equations, he studied the structure of a general expression built with radicals and he asked under which conditions such an expression was the root of an algebraic equation of given degree. In the theory of Abelian integrals, he investigated the most general algebraic relation between given integrals and he proved that it is reducible to a linear relation. In the case of elliptic functions, a further reduction led to complex multiplication. This part of Abel's work announces Liouville's investigations on integration in finite terms and his classification of transcendental functions. We saw the same concern with generality in Abel's treatment of functional equations. This general method of Abel is well ahead of his time and close to the modern conception of axiomatic method.

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