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Jean-Pierre Serre

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Jean-Pierre Serre: Mon premier demi-siècle au Collège de France

Jean-Pierre Serre: My First Fifty Years at the Collège de France

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Jean-Pierre Serre, Professeur au Collège de France, titulaire de la chaire d'*Algèbre et Géométrie* de 1956 à 1994.

Vous avez enseigné au Collège de France de 1956 à 1994, dans la chaire d'Algèbre et Géométrie. Quel souvenir en gardez-vous?

J'ai occupé cette chaire pendant 38 ans. C'est une longue période, mais il y a des précédents: si l'on en croit l'Annuaire du Collège de France, au XIX^e siècle, la chaire de physique n'a été occupée que par deux professeurs: l'un est resté 60 ans, l'autre 40. Il est vrai qu'il n'y avait pas de retraite à cette époque et que les professeurs avaient des suppléants (auxquels ils versaient une partie de leur salaire).

Quant à mon enseignement, voici ce que j'en disais dans une interview de 1986[1]: "Enseigner au Collège est un privilège merveilleux et redoutable. Merveilleux à cause de la liberté dans le choix des sujets et du haut niveau de l'auditoire: chercheurs au CNRS, visiteurs étrangers, collègues de Paris et d'Orsay — beaucoup sont des habitués qui viennent régulièrement depuis cinq, dix ou même vingt ans. Redoutable aussi: il faut chaque année un sujet de cours nouveau, soit sur ses propres recherches (ce que je préfère), soit sur celles des autres; comme un cours annuel dure environ vingt heures, cela fait beaucoup!"

Comment s'est passée votre leçon inaugurale?

À mon arrivée au Collège, j'étais un jeune homme de trente ans. La leçon inaugurale m'apparaissait presque comme un oral d'examen, devant professeurs, famille, collègues mathématiciens, journalistes, etc. J'ai essayé de la préparer. Au bout d'un mois, j'avais réussi à en écrire une demi-page.

Arrive le jour de la leçon, un moment assez solennel. J'ai commencé par lire la demi-page en question, puis j'ai improvisé. Je ne sais plus très bien ce que j'ai dit (je me souviens seulement avoir parlé de l'Algèbre, et du rôle ancillaire qu'elle joue en Géométrie et en Théorie des Nombres). D'après le compte-rendu paru dans le journal *Combat*, j'ai passé mon temps à essuyer machinalement la table qui me séparait du public; je ne me suis senti à l'aise que lorsque j'ai pris en main un bâton de craie et que j'ai commencé à écrire sur le tableau noir, ce vieil ami des mathématiciens.

Quelques mois plus tard, le secrétariat m'a fait remarquer que toutes les leçons inaugurales étaient rédigées et que la mienne ne l'était pas. Comme elle avait été improvisée, j'ai proposé de la recommencer dans le même style, en me remettant mentalement dans la même situation. Un beau soir, on m'a ouvert un bureau du Collège et l'on m'a prêté un magnétophone. Je me suis efforcé de recréer l'atmosphère initiale, et j'ai refait une leçon sans doute à peu près semblable à l'originale. Le lendemain, j'ai apporté le magnétophone au secrétariat; on m'a dit que l'enregistrement était inaudible. J'ai estimé que j'avais fait tout mon possible et je m'en suis tenu là. Ma leçon inaugurale est restée la seule qui n'ait jamais été rédigée.

En règle générale, je n'écris pas mes exposés; je ne consulte pas mes notes (et, souvent, je n'en ai pas). J'aime réfléchir devant mes auditeurs. J'ai le sentiment, Jean-Pierre Serre, Professor at the Collège de France, held the Chair in *Algebra and Geometry* from 1956 to 1994.

You taught at the Collège de France from 1956 to 1994, holding the Chair in Algebra and Geometry. What are you memories of your time there?

I held the Chair for 38 years. That is a long time, but there were precedents. According to the Yearbook of the Collège de France, the Chair in Physics was held by just two professors in the 19th century: one remained in his post for 60 years, and the other for 40. It is true that there was no retirement in that era and that professors had deputies (to whom they paid part of their salaries).

As for my teaching career, this is what I said in an interview in 1986[1]: "Teaching at the Collège is both a marvelous and a challenging privilege. Marvelous because of the freedom of choice of subjects and the high level of the audience: CNRS [Centre national de la recherche scientifique] researchers, visiting foreign academics, colleagues from Paris and Orsay—many regulars who have been coming for 5, 10 or even 20 years. It is challenging too: new lectures have to be given each year, either on one's own research (which I prefer), or on the research of others. Since a series of lectures for a year's course is about 20 hours, that's quite a lot!"

Can you tell us about your inaugural lecture?

I was a young man, about 30, when I arrived at the Collège. The inaugural lecture was almost like an oral examination in front of professors, family, mathematician colleagues, journalists, etc. I tried to prepare it, but after a month I had only managed to write half a page.

When the day of the lecture came, it was quite a tense moment. I started by reading the half page I had prepared and then I improvised. I can no longer remember what I said (I only recall that I spoke about algebra and the ancillary role it plays in geometry and number theory). According to the report that appeared in the newspaper *Combat*, I spent most of the time mechanically wiping the table that separated me from my audience. I did not feel at ease until I had a piece of chalk in my hand and I started to write on the blackboard, the mathematician's old friend.

A few months later, the Secretary's Office told me that all inaugural lectures were written up, but they had not received the transcript of mine. As it had been improvised, I offered to repeat it in the same style, mentally putting myself back in the same situation. One evening, I was given a tape recorder and I went into an office at the Collège. I tried to recall the initial atmosphere, and to make up a lecture as close as possible to the original one. The next day I returned the tape recorder to the Secretary's Office. They told me that the recording was inaudible. I decided that I had done all I could and left it there. My inaugural lecture is still the only one that has not been written up.

As a rule, I don't write my lectures. I don't consult notes (and often I don't have any). I like to do my thinking in front of the audience. When I am explaining lorsque j'explique des mathématiques, de parler à un ami. Devant un ami, on n'a pas envie de lire un texte. Si l'on a oublié une formule, on en donne la structure; cela suffit. Pendant l'exposé j'ai en tête une quantité de choses qui me permettraient de parler bien plus longtemps que prévu. Je choisis suivant l'auditoire, et l'inspiration du moment.

Seule exception: le séminaire Bourbaki, où l'on doit fournir un texte suffisamment à l'avance pour qu'il puisse être distribué en séance. C'est d'ailleurs le seul séminaire qui applique une telle règle, très contraignante pour les conférenciers.

Quel est la place de Bourbaki dans les mathématiques françaises d'aujourd'hui?

C'est le séminaire qui est le plus intéressant. Il se réunit trois fois par an, en mars, mai et novembre. Il joue un rôle à la fois social (occasion de rencontres) et mathématique (exposé de résultats récents — souvent sous une forme plus claire que celle des auteurs); il couvre toutes les branches des mathématiques.

Les livres (*Topologie, Algèbre, Groupes de Lie,...*) sont encore lus, non seulement en France, mais aussi à l'étranger. Certains de ces livres sont devenus des classiques: je pense en particulier à celui sur les systèmes de racines. J'ai vu récemment (dans le *Citations Index* de l'AMS[2]) que Bourbaki venait au 6^{e} rang (par nombre de citations) parmi les mathématiciens français (de plus, au niveau mondial, les n^{os} 1 et 3 sont des Français, et s'appellent tous deux Lions: un bon point pour le Collège). J'ai gardé un très bon souvenir de ma collaboration à Bourbaki, entre 1949 et 1973. Elle m'a appris beaucoup de choses, à la fois sur le fond (en me forçant à rédiger des choses que je ne connaissais pas) et sur la forme (comment écrire de façon à être compris). Elle m'a appris aussi à ne pas trop me fier aux "spécialistes."

La méthode de travail de Bourbaki est bien connue: distribution des rédactions aux différents membres et critique des textes par lecture à haute voix (ligne à ligne: c'est lent mais efficace). Les réunions (les "congrès") avaient lieu 3 fois par an. Les discussions étaient très vives, parfois même passionnées. En fin de congrès, on distribuait les rédactions à de nouveaux rédacteurs. Et l'on recommençait. Le même chapitre était souvent rédigé quatre ou cinq fois. La lenteur du processus explique que Bourbaki n'ait publié finalement qu'assez peu d'ouvrages en quarante années d'existence, depuis les années 1930–1935 jusqu'à la fin des années 1970, où la production a décliné.

En ce qui concerne les livres eux-mêmes, on peut dire qu'ils ont rempli leur mission. Les gens ont souvent cru que ces livres traitaient des sujets que Bourbaki trouvait intéressants. La réalité est différente: ses livres traitent de ce qui est utile pour faire des choses intéressantes. Prenez l'exemple de la théorie des nombres. Les publications de Bourbaki en parlent très peu. Pourtant, ses membres l'appréciaient beaucoup, mais ils jugeaient que cela ne faisait pas partie des *Éléments*: il fallait d'abord avoir compris beaucoup d'algèbre, de géométrie et d'analyse.

Par ailleurs, on a souvent imputé à Bourbaki tout ce que l'on n'aimait pas en mathématiques. On lui a reproché notamment les excès des "maths modernes" dans les programmes scolaires. Il est vrai que certains responsables de ces programmes se mathematics, I feel I am speaking to a friend. You don't want to read a text out to a friend; if you have forgotten a formula, you give its structure; that's enough. During the lecture I have a lot of possible material in my mind—much more than possible in the allotted time. What I actually say depends on the audience and my inspiration.

Only exception: the Bourbaki seminar for which one has to provide a text sufficiently in advance so that it can be distributed during the meeting. This is the only seminar that applies this rule; it is very restrictive for lecturers.

What is Bourbaki's place in French mathematics now?

Its most interesting feature is the Bourbaki seminar. It is held three times a year, in March, May and November. It plays both a social role (an occasion for meeting other people) and a mathematical one (the presentation of recent results—often in a form that is clearer than that given by the authors). It covers all branches of mathematics.

Bourbaki's books (*Topology, Algebra, Lie Groups*, etc.) are still widely read, not just in France but also abroad. Some have become classics: I'm thinking in particular about the book on root systems. I recently saw (in the AMS *Citations Index*[2]) that Bourbaki ranked sixth (by number of citations) among French mathematicians. (What's more, at the world level, numbers 1 and 3 are French and both are called Lions: a good point for the Collège.) I have very good memories of my collaboration with Bourbaki from 1949 to 1973. Bourbaki taught me many things, both on background (making me write about things which I did not know very well) and on style (how to write in order to be understood). Bourbaki also taught me not to rely on "specialists".

Bourbaki's working method is well-known: the distribution of drafts to the various members and their criticism by reading them aloud (line by line: slow but effective). The meetings ("congrès") were held three times a year. The discussions were very lively, sometimes passionate. At the end of each congrès, the drafts were distributed to new writers. And so on. A chapter could often be written four or five times. The slow pace of the process explains why Bourbaki ended up publishing with relatively few books over the 40 years from 1930–1935 till the end of the 1970s when production faded away.

As for the books themselves, one may say that they have fulfilled their mission. People often believe that these books deal with subjects that Bourbaki found interesting. The reality is different: the books deal with what is useful in order to do interesting things. Take number theory for example. Bourbaki's publications hardly mention it. However, the Bourbaki members liked it very much it, but they considered that it was not part of the *Elements*: it needed too much algebra, geometry and analysis.

Besides, Bourbaki is often blamed for everything that people do not like about mathematics, especially the excesses of "modern math" in school curricula. It is true that some of those responsible for these curricula claimed to follow Bourbaki. But sont réclamés de Bourbaki. Mais Bourbaki n'y était pour rien: ses écrits étaient destinés aux mathématiciens, pas aux étudiants, encore moins aux adolescents. Notez que Bourbaki a évité de se prononcer sur ce sujet. Sa doctrine était simple: on fait ce que l'on choisit de faire, on le fait du mieux que l'on peut, mais on n'explique pas pourquoi on le fait. J'aime beaucoup ce point de vue qui privilégie le travail par rapport au discours — tant pis s'il prête parfois à des malentendus.

Comment analysez-vous l'évolution de votre discipline depuis l'époque de vos débuts? Est-ce que l'on fait des mathématiques aujourd'hui comme on les faisait il y a cinquante ans?

Bien sûr, on fait des mathématiques aujourd'hui comme il y a cinquante ans! Évidemment, on comprend davantage de choses; l'arsenal de nos méthodes a augmenté. Il y a un progrès continu. (Ou parfois un progrès par à-coups: certaines branches restent stagnantes pendant une décade ou deux, puis brusquement se réveillent quand quelqu'un introduit une idée nouvelle.)

Si l'on voulait dater les mathématiques “modernes” (un terme bien dangereux), il faudrait sans doute remonter aux environs de 1800 avec Gauss.

Et en remontant plus loin, si vous rencontriez Euclide, qu'auriez-vous à vous dire?

Euclide me semble être plutôt quelqu'un qui a mis en ordre les mathématiques de son époque. Il a joué un rôle analogue à celui de Bourbaki il y a cinquante ans. Ce n'est pas par hasard que Bourbaki a choisi d'intituler ses ouvrages des *Éléments de Mathématique*: c'est par référence aux *Éléments* d'Euclide. (Notez aussi que “Mathématique” est écrit au singulier. Bourbaki nous enseigne qu'il n'y a pas plusieurs mathématiques distinctes, mais une seule mathématique. Et il nous l'enseigne à sa façon habituelle: pas par de grands discours, mais par l'omission d'une lettre à la fin d'un mot.)

Pour en revenir à Euclide, je ne pense pas qu'il ait produit des contributions réellement originales. Archimède serait un interlocuteur plus indiqué. C'est lui le grand mathématicien de l'Antiquité. Il a fait des choses extraordinaires, aussi bien en mathématique qu'en physique.

En philosophie des sciences, il y a un courant très fort en faveur d'une pensée de la rupture. N'y a-t-il pas de ruptures en mathématiques? On a décrit par exemple l'émergence de la probabilité comme une manière nouvelle de se représenter le monde. Quelle est sa signification en mathématiques?

Les philosophes aiment bien parler de “rupture.” Je suppose que cela ajoute un peu de piment à leurs discours. Je ne vois rien de tel en mathématique: ni catastrophe, ni révolution. Des progrès, oui, je l'ai déjà dit; ce n'est pas la même chose. Nous travaillons tantôt à de vieilles questions, tantôt à des questions nouvelles. Il n'y a pas de frontière entre les deux. Il y a une grande continuité entre les mathématiques

Bourbaki had nothing to do with it: its books are meant for mathematicians, not for students, and even less for teen-agers. Note that Bourbaki was careful not to write anything on this topic. Its doctrine was simple: one does what one chooses to do, one does it the best one can, but one does not explain why. I very much like this attitude which favors work over discourse—too bad if it sometimes lead to misunderstandings.

How would you describe the development of your discipline since the time when you were starting out? Is mathematics conducted nowadays as it was 50 years ago?

Of course you do mathematics today like 50 years ago! Clearly more things are understood; the range of our methods has increased. There is continuous progress. (Or sometimes leaps forward: some branches remain stagnant for a decade or two and then suddenly there's a reawakening as someone introduces a new idea.)

If you want to put a date on "modern" mathematics (a very dangerous term), you would have to go back to about 1800 and Gauss.

Going back further, if you were to meet Euclid, what would you say to him?

Euclid seems to me like someone who just put the mathematics of his era into order. He played a role similar to Bourbaki's 50 years ago. It is no coincidence that Bourbaki decided to give its treatise the title *Éléments de Mathématique*. This is a reference to Euclid's *Éléments*. (Note that "Mathématique" is written in the singular. Bourbaki tells us that rather than several different mathematics there is one single mathematics. And he tells us in his usual way: not by a long discourse, but by the omission of one letter from the end of one word.)

Coming back to Euclid, I don't think that he came up with genuinely original contributions. Archimedes would be much more interesting to talk to. He was the great mathematician of antiquity. He did extraordinary things, both in mathematics and physics.

In the philosophy of science there is a very strong current in favor of the concept of rupture. Are there ruptures in mathematics? For example the emergence of probability as a new way in which to represent the world. What is its significance in mathematics?

Philosophers like to talk of "rupture". I suppose it adds a bit of spice to what they say. I do not see anything like that in mathematics: no catastrophe and no revolution. Progress, yes, as I've already said; but that is not the same. We work sometimes on old questions and sometimes on new ones. There is no boundary between the two. There is a deep continuity between the mathematics of two centuries ago and that d'il y a deux siècles et celles de maintenant. Le temps des mathématiciens est la "longue durée" de feu mon collègue Braudel.

Quant aux probabilités, elles sont utiles pour leurs applications à la fois mathématiques et pratiques; d'un point de vue purement mathématique, elles constituent une branche de la théorie de la mesure. Peut-on vraiment parler à leur sujet de "manière nouvelle de se représenter le monde"? Sûrement pas en mathématique.

Est-ce que les ordinateurs changent quelque chose à la façon de faire des mathématiques?

On avait coutume de dire que les recherches en mathématiques étaient peu coûteuses: des crayons et du papier, et voilà nos besoins satisfaits. Aujourd'hui, il faut ajouter les ordinateurs. Cela reste peu onéreux, dans la mesure où les mathématiciens ont rarement besoin de ressources de calcul très importantes. À la différence, par exemple, de la physique des particules, dont les besoins en calcul sont à la mesure des très grands équipements nécessaires au recueil des données, les mathématiciens ne mobilisent pas de grands centres de calcul.

En pratique, l'informatique change les conditions matérielles du travail des mathématiciens: on passe beaucoup de temps devant son ordinateur. Il a différents usages. Tout d'abord, le nombre des mathématiciens a considérablement augmenté. À mes débuts, il y a 55 ou 60 ans, le nombre des mathématiciens productifs était de quelques milliers (dans le monde entier), l'équivalent de la population d'un village. À l'heure actuelle, ce nombre est d'au moins 100 000: une ville. Cet accroissement a des conséquences pour la manière de se contacter et de s'informer. L'ordinateur et Internet accélèrent les échanges. C'est d'autant plus précieux que les mathématiciens ne sont pas ralentis, comme d'autres, par le travail expérimental: nous pouvons communiquer et travailler très rapidement. Je prends un exemple. Un mathématicien a trouvé une démonstration mais il lui manque un lemme de nature technique. Au moyen d'un moteur de recherche — comme Google — il repère des collègues qui ont travaillé sur la question et leur envoie un e-mail. De cette manière, il a toutes les chances de trouver en quelques jours ou même en quelques heures la personne qui a effectivement démontré le lemme dont il a besoin. (Bien entendu, ceci ne concerne que des problèmes auxiliaires: des points de détail pour lesquels on désire renvoyer à des références existantes plutôt que de refaire soi-même les démonstrations. Sur des questions vraiment difficiles, mon mathématicien aurait peu de chances de trouver quelqu'un qui puisse lui venir en aide.)

L'ordinateur et Internet sont donc des outils d'accélération de notre travail. Ils permettent aussi de rendre les manuscrits accessibles dans le monde entier, sans attendre leur parution dans un journal. C'est très pratique. Notez que cette accélération a aussi des inconvénients. Le courrier électronique produit des correspondances informelles que l'on conserve moins volontiers que le papier. On jette rarement des lettres alors que l'on efface ou l'on perd facilement les emails (quand on change d'ordinateur, par exemple). On a publié récemment (en version bilingue: français sur une page, et anglais sur la page d'en face) ma correspondance avec A. Grothendieck entre 1955 et 1987; cela n'aurait pas été possible si elle avait été électronique.

of today. The time of mathematicians is the “longue durée” of my late colleague the historian Fernand Braudel.

As for probability theory, it is useful for its applications both to mathematics and to practical questions From a purely mathematical point of view, it is a branch of measure theory. Can one really describe it as “a new way in which to represent the world”? Surely not in mathematics.

Have computers changed the manner in which mathematics is conducted?

It used to be said that mathematical research was cheap: paper and pencils, that was all we needed. Nowadays, you have to add computers. It is not very expensive, since mathematicians rarely need a lot of processing power. This is different from, say, particle physics, where a lot of equipment is required.

In practice, computers have changed the material conditions of mathematicians’ work: we spend a lot of time in front of our computer. It has several different uses. First of all, there are now considerably more mathematicians. When I started out, some 55 or 60 years ago, there were only a few thousand productive mathematicians (in the whole world), the equivalent of a village. Now, this number has grown to at least 100 000: a city. This growth has consequences for the way mathematicians contact each other and gain information. The computer and Internet have accelerated exchanges. This is especially important for us, since we are not slowed down, as others, by experimental work: we can communicate and work very rapidly. Let me give you an example. If a mathematician is working on a proof but needs a technical lemma, then through a search engine—such as Google—he will track down colleagues who have worked on the question and send them an e-mail. In this way, in just a few days or even hours, he may be able to find somebody who has proved the required lemma. (Of course, this only applies to easy problems: those for which you want to use a reference rather than to reconstruct a proof. For really difficult questions, a mathematician would have little chance of finding someone to help him.)

Computer and Internet are thus the tools which speed up our work. They allow us to make our manuscripts accessible to everybody without waiting for publication in a journal. That is very convenient. But this acceleration also has its disadvantages. E-mail produces informal correspondence which is less likely to be kept than the paper one. It is unusual to throw letters away but one can easily delete or lose e-mails (when one changes computers for example). Recently a bilingual version (French on one page and English on the other) of my correspondence with A. Grothendieck between 1955 and 1987 has been published. That would not have been possible if the correspondence had been by e-mail.

Par ailleurs, certaines démonstrations font appel à l'ordinateur pour vérifier une série de cas qu'il serait impraticable de traiter à la main. Deux cas classiques: le problème des 4 couleurs (coloriage des cartes avec seulement quatre couleurs) et le problème de Képler (empilement des sphères dans l'espace à 3 dimensions). Cela conduit à des démonstrations qui ne sont pas réellement vérifiables; autrement dit, ce ne sont pas de vraies "démonstrations" mais seulement des faits expérimentaux, très vraisemblables, mais que personne ne peut garantir.

Vous avez évoqué l'augmentation du nombre des mathématiciens. Quelle est aujourd'hui la situation. Où vont les mathématiques?

L'augmentation du nombre des mathématiciens est un fait important. On pouvait craindre que cela se fasse au détriment de la qualité. En fait, il n'y a rien eu de tel. Il y a beaucoup de très bons mathématiciens (en particulier parmi les jeunes français — un très bon augure).

Ce que je peux dire, concernant l'avenir, c'est qu'en dépit de ce grand nombre de mathématiciens, nous ne sommes pas à court de matière. Nous ne manquons pas de problèmes, alors qu'il y a un peu plus de deux siècles, à la fin du XVIII^{e}, Lagrange était pessimiste: il pensait que "la mine était tarie," qu'il n'y avait plus grand-chose à trouver. Lagrange a écrit cela juste avant que Gauss ne relance les mathématiques de manière extraordinaire, à lui tout seul. Aujourd'hui, il y a beaucoup de terrains à prospecter pour les jeunes mathématiciens (et aussi pour les moins jeunes, j'espère).

Selon un lieu commun de la philosophie des sciences, les grandes découvertes mathématiques sont le fait de mathématiciens jeunes. Est-ce votre cas?

Je ne crois pas que le terme de "grande découverte" s'applique à moi. J'ai surtout fait des choses "utiles" (pour les autres mathématiciens). En tout cas, lorsque j'ai eu le prix Abel en 2003, la plupart des travaux qui ont été cités par le jury avaient été faits avant que je n'aie 30 ans. Mais si je m'étais arrêté à ce moment-là, on ne m'aurait sans doute pas donné ce prix: j'ai fait aussi d'autres choses par la suite (ne serait-ce que des "conjectures" sur lesquelles beaucoup de gens ont travaillé et travaillent encore).

Dans ma génération, plusieurs de mes collègues ont continué au-delà de 80 ans, par exemple mes vieux amis Armand Borel et Raoul Bott, morts tous deux récemment à 82 ans. Il n'y a pas de raison de s'arrêter, tant que la santé le permet. Encore faut-il que le sujet s'y prête. Quand on a des sujets très larges, il y a toujours quelque chose à faire, mais si l'on est trop spécialisé on peut se retrouver bloqué pendant de longues périodes, soit parce que l'on a démontré tout ce qu'il y avait à démontrer, soit au contraire parce que les problèmes sont trop difficiles. C'est très frustrant.

Les découvertes mathématiques donnent de grandes joies. Poincaré, Hadamard, Littlewood[3] l'ont très bien expliqué. En ce qui me concerne, je garde surtout le souvenir d'une idée qui a contribué à débloquer la théorie de l'homotopie. Cela s'est passé une nuit de retour de vacances, en 1950, dans une couchette de train. Je cherchais un espace fibré ayant telles et telles propriétés. La réponse est venue:

On the other hand, some proofs do need a computer in order to check a series of cases that would be impossible to do by hand. Two classic examples are the four-color problem (shading maps using only four colors) and the Kepler conjecture (packing spheres into three-dimensional space). This leads to proofs which are not really verifiable; in other words, they are not genuine "proofs" but just experimental facts, very plausible, but nobody can guarantee them.

You mentioned the increasing number of mathematicians today. But where is mathematics going?

The increase in the number of mathematicians is an important fact. One could have feared that this increase in size was to the detriment of quality. But in fact, this is not the case. There are many very good mathematicians (in particular young French mathematicians—a good omen for us).

What I can say about the future is that, despite this huge number of mathematicians, we are not short of subject matter. There is no lack of problems, even though just two centuries ago, at the end of the 18th century, Lagrange was pessimistic: he thought that "the mine was exhausted", and that there was nothing much more to discover. Lagrange wrote this just before Gauss relaunched mathematics in an extraordinary way, all by himself. Today, there are many fields to explore for young mathematicians (and even for those who are not so young, I hope).

It is often said in the philosophy of science that major mathematical discoveries are made by young mathematicians. Was this the case for you?

I don't believe that the term "major discovery" applies to me. I have rather done things that are "useful" (for other mathematicians). When I was awarded the Abel prize in 2003, most of the work cited by the jury had been done before I was 30. But if I had stopped then, it would probably not have awarded me the prize. I have done other things after that (if only some conjectures that have kept many people busy).

Of my generation, several of my colleagues have continued working beyond the age of 80. For example, my old friends Armand Borel and Raoul Bott, who both recently died aged 82. There is no reason to stop, as long as health allows it. But the subject matter has to be there. When you are dealing with very broad subjects, there is always something to do, but if you are too specialized you can find yourself blocked for long periods of time, either because you have proved everything that can be proved, or, to the contrary, because the problems are too difficult. It is very frustrating.

Discoveries in mathematics can bring great joy. Poincaré, Hadamard and Littlewood[3] have explained it very well. As for myself, I still have the memory of an idea that contributed to unlocking homotopy theory. It happened one night while traveling home from vacation in 1950 in the sleeping car of a train. I had been looking for a fiber space with such and such properties. Then the answer came:

l'espace des lacets! Je n'ai pas pu m'empêcher de réveiller ma femme qui dormait dans la couchette du dessous pour lui dire: ça y est! Ma thèse est sortie de là, et bien d'autres choses encore. Bien sûr, ces découvertes soudaines sont rares: cela m'est arrivé peut-être deux fois en soixante ans. Mais ce sont des moments lumineux, vraiment exceptionnels.

Le Collège de France est-il un endroit où l'on échange avec d'autres disciplines?

Non, pas pour moi. Même entre les mathématiciens du Collège, il n'y a pas de travail collectif. Il faut préciser que nous travaillons dans des branches souvent très séparées. Ce n'est pas un mal: le Collège n'est pas censé être un club. Un certain nombre de lieux communs modernes — comme le *travail collectif*, l'*interdisciplinarité* et le *travail en équipe* — ne s'appliquent pas à nous.

Qu'avez-vous pensé du dialogue entre un spécialiste de neurosciences, Jean-Pierre Changeux, et le mathématicien Alain Connes, qui est restitué dans le livre Matière à pensée*?*

Ce livre est un bel exemple de dialogue de sourds. Changeux ne comprend pas ce que dit Connes, et inversement. C'est assez étonnant. Personnellement, je suis du côté de Connes. Les vérités mathématiques sont indépendantes de nous[4]. Notre seul choix porte sur la façon de les exprimer. Si on le désire, on peut le faire sans introduire aucune terminologie. Considérons par exemple une troupe de soldats. Leur général aime les arranger de deux façons, soit en rectangle, soit en 2 carrés. C'est au sergent de les placer. Il s'aperçoit qu'il n'a qu'à les mettre en rang par 4: s'il en reste 1 qu'il n'a pas pu placer, ou bien il arrivera à les mettre tous en rectangle, ou bien il arrivera à les répartir en deux carrés.

[Traduction technique: le nombre n des soldats est de la forme $4k + 1$. Si n n'est pas premier, on peut arranger les soldats en rectangle. Si n est premier, un théorème dû à Fermat dit que n est somme de deux carrés.]

Quelle est la place des mathématiques par rapport aux autres sciences? Y a-t-il une demande nouvelle de mathématiques, venant de ces sciences?

Sans doute, mais il faut séparer les choses. Il y a d'une part la physique théorique, qui est tellement théorique qu'elle est à cheval entre mathématique et physique, les physiciens considérant que ce sont des mathématiques, tandis que les mathématiciens sont d'un avis contraire. Elle est symbolisée par la théorie des cordes. Son aspect le plus positif est de fournir aux mathématiciens un grand nombre d'énoncés, qu'il leur faut démontrer (ou éventuellement démolir).

Par ailleurs, notamment en biologie, il y a tout ce qui relève de systèmes comportant un grand nombre d'éléments qu'il faut traiter collectivement. Il existe des branches des mathématiques qui s'occupent de ces questions. Cela répond à une demande. Il y a aussi des demandes qui concernent la logique: c'est le cas de the loop space! I couldn't help from waking up my wife who was sleeping in the bunk below: "I've got it!" I said. My thesis, and many other things, originated from that idea. Of course, these sudden discoveries are rare: they have only happened to me twice in sixty years. But they are illuminating moments: truly exceptional.

Are there exchanges between the disciplines at the Collège de France?

No, not for me. There is no collective work even between the mathematicians at the Collège. We work on quite different things. This is not a bad thing. The Collège is not supposed to be a club. Many commonplace sayings, such as *collective work*, *interdisciplinarity* and *team work*, do not apply to us.

What do you think about the dialogue between the neuroscientist Jean-Pierre Changeux and the mathematician Alain Connes, recorded in the book "Matière à pensée"?

This book is a good example of dialogue of the deaf. Changeux does not understand what Connes says and vice versa. It is quite astonishing. Personally, I am on Connes' side. Mathematical truths are independent of us.[4] Our only choice is in the way in which we express them. If you want, you can do this without introducing any terminology. Consider, for example, a company of soldiers. The general likes to arrange them in two ways, either in a rectangle or in two squares. It is up to the sergeant to put them in the correct positions. He realizes that he only has to put them in rows of four: if there is one left over that he cannot place, either he will manage to put them all in a rectangle, or manage to arrange them in two squares.

[Technical translation: the number n of soldiers is congruent to 1 (mod 4). If n is not a prime, the soldiers can be arranged in a rectangle. If n is a prime, a theorem of Fermat shows that n is the sum of two squares.]

What is the place of mathematics in relation to other sciences? Is there a renewed demand for mathematics from these sciences?

Probably, but there are different cases. Some theoretical physics is so theoretical that it is half way between mathematics and physics. Physicists consider it mathematics, while mathematicians have the opposite view. String theory is a good example. The most positive aspect is to provide mathematicians with a large number of statements which they have to prove (or maybe disprove).

On the other hand, in particular in biology, there are situations involving very many elements that have to be processed collectively. There are branches of mathematics that deal with such questions. They meet a need. Another branch, logic, is l'informatique, pour la fabrication des ordinateurs. Il faut mentionner aussi la cryptographie, qui est une source de problèmes intéressants relatifs à la théorie des nombres.

En ce qui concerne la place des mathématiques par rapport aux autres sciences, on peut voir les mathématiques comme un grand entrepôt empli de rayonnages. Les mathématiciens déposent sur les rayons des choses dont ils garantissent qu'elles sont vraies; ils en donnent aussi le mode d'emploi et la manière de les reconstituer. Les autres sciences viennent se servir en fonction de leurs besoins. Le mathématicien ne s'occupe pas de ce qu'on fait de ses produits. Cette métaphore est un peu triviale, mais elle reflète assez bien la situation. (Bien entendu, on ne choisit pas de faire des mathématiques pour mettre des choses sur les rayons: on fait des mathématiques pour le plaisir d'en faire.)

Voici un exemple personnel. Ma femme, Josiane, était spécialiste de chimie quantique. Elle avait besoin d'utiliser les représentations linéaires de certains groupes de symétries. Les ouvrages disponibles n'étaient pas satisfaisants: ils étaient corrects, mais employaient des notations très lourdes. J'ai rédigé pour elle un exposé adapté à ses besoins, et je l'ai ensuite publié dans un livre intitulé *Représentations Linéaires des Groupes Finis*. J'ai fait mon travail de mathématicien (et de mari): mis des choses sur les rayons.

Le vrai en mathématiques a-t-il le même sens qu'ailleurs?

Non. C'est un vrai absolu. C'est sans doute ce qui fait l'impopularité des mathématiques dans le public. L'homme de la rue veut bien tolérer l'absolu quand il s'agit de religion, mais pas quand il s'agit de mathématique. Conclusion: croire est plus facile que démontrer.

useful for the building of computers. Cryptography should also be mentioned; it is a source of interesting problems in number theory.

As for the place of mathematics in relation to other sciences, mathematics can be seen as a big warehouse full of shelves. Mathematicians put things on the shelves and guarantee that they are true. They also explain how to use them and how to reconstruct them. Other sciences come and help themselves from the shelves; mathematicians are not concerned with what they do with what they have taken. This metaphor is rather coarse, but it reflects the situation well enough. (Of course one does not choose to do mathematics just for putting things on shelves; one does mathematics for the fun of it.)

Here is a personal example. My wife, Josiane, was a specialist in quantum chemistry. She needed linear representations of certain symmetry groups. The books she was working with were not satisfactory; they were correct, but they used very clumsy notation. I wrote a text that suited her needs, and then published it in book form, as *Linear Representations of Finite Groups*. I thus did my duty as a mathematician (and as a husband): putting things on the shelves.

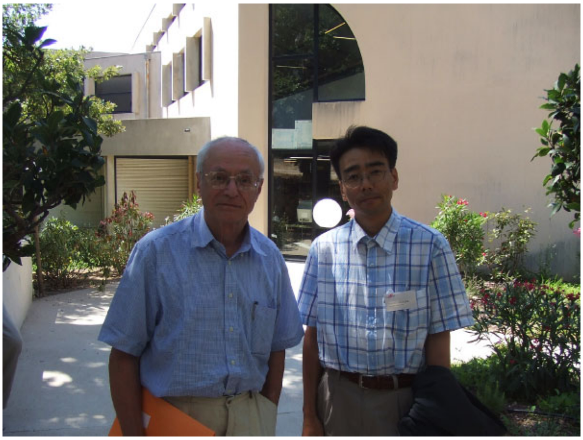
Does truth in mathematics have the same meaning as elsewhere?

No. It's an absolute truth. This is probably what makes mathematics unpopular with the public. The man in the street accepts the absolute in religion, but not in mathematics. Conclusion: to believe is easier than to prove.

1 M. Schmidt, *Hommes de Science*, 218–227, Hermann, Paris, 1990.
2 AMS: American Mathematical Society.
3 J. E. Littlewood, *A Mathematician's Miscellany*, Methuen and Co, 1953. Ce livre explique bien la part inconsciente du travail créatif.
4 Il y a quelques années, mon ami R. Bott et moi-même allions recevoir un prix israélien (le prix Wolf) remis dans la Knesset, à Jerusalem. Bott devait dire quelques mots sur les mathématiques. Il m'a demandé: que dire? Je lui ai dit "C'est bien simple; tu n'as qu'à expliquer ceci: les autres sciences cherchent à trouver les lois que Dieu a choisies; les mathématiques cherchent à trouver les lois auxquelles Dieu a dû obéir." C'est ce qu'il a dit. La Knesset a apprécié.
1· M. Schmidt, *Hommes de Science*, 218–227, Hermann, Paris, 1990.
2· AMS: American Mathematical Society
3· J.E. Littlewood, *A Mathematician's Miscellany*, Methuen and Co., 1953. This book offers a very good description of the unconscious aspect of creative work.
4· A few years ago, my friend R. Bott and myself went to receive a prize in Israel (the Wolf prize) awarded by the Knesset in Jerusalem. Bott had to say a few words on mathematics. He asked me what he should say. I replied: "It's very simple, all you have to explain is this: other sciences seek to discover the laws that God has chosen; mathematics seeks to discover the laws which God has to obey". And that is what he said. The Knesset appreciated it.

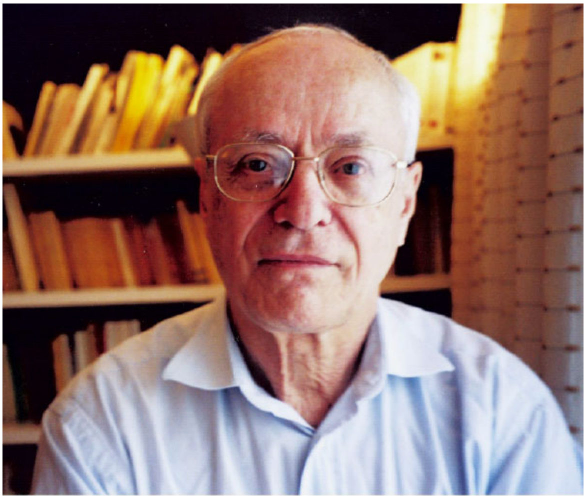
Serre and Henri Cartan, Prix Julia 1970

Anatole Abragam, Serre, and Jaques Tits

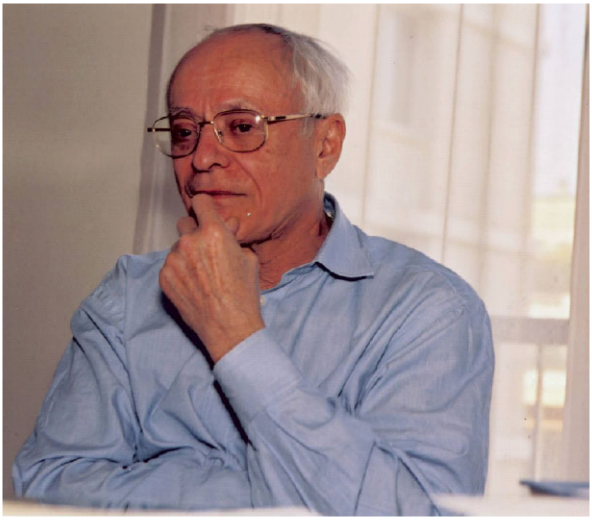


Serre and Yuichiro Taguchi

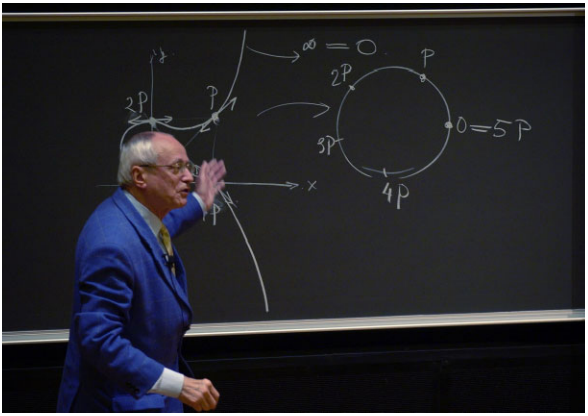
Serre



Serre, May 9, 2003 (photo by Chino Hasebe)



Serre, 2003



The Abel Lecture, Oslo 2003

Jean-Pierre Serre: An Overview of His Work

Pilar Bayer

Introduction

The work of Jean-Pierre Serre represents an important breakthrough in at least four mathematical areas: algebraic topology, algebraic geometry, algebra, and number theory. His outstanding mathematical achievements have been a source of inspiration for many mathematicians. His contributions to the field were recognized in 2003 when he was awarded the Abel Prize by the Norwegian Academy of Sciences and Letters, presented on that occasion for the first time.

To date, four volumes of Serre's work [*Œuvres, Collected Papers* I–II–III(1986); IV(2000)] ([S210], [S211], [S212]; [S261]) have been published by Springer-Verlag. These volumes include 173 papers, from 1949 to 1998, together with comments on later developments added by the author himself. Some of the papers that he coauthored with A. Borel are to be found in [A. Borel. *Œuvres*, *Collected Papers*. Springer, 1983]. Serre has written some twenty books, which have been frequently reprinted and translated into several languages (mostly English and Russian, but sometimes also Chinese, German, Japanese, Polish or Spanish). He has also delivered lectures in many seminars: Bourbaki, Cartan, Chevalley, Delange–Pisot–Poitou, Grothendieck, Sophus Lie, etc.; some of them have been gathered in the books [S262, SEM(2001; 2008)].

Summarizing his work is a difficult task—especially because his papers present a rich web of interrelationships and hence can hardly be put in linear order. Here, we have limited ourselves to a presentation of their contents, with only a brief discussion of their innovative character.

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Broadly speaking, the references to his publications are presented thematically and chronologically. In order to facilitate their location, the number of a paper corresponding to the present List of Publications is followed by its number in the *Œuvres* (if applicable) and by the year of its publication. Thus, a quotation of the form [S216, Œ 143(1987)] will refer to paper 216 of the List included in the *Œuvres* as number 143. Serre's books will be referred to in accordance with the List and the References at the end of this manuscript. The names of other authors, followed by a date, will denote the existence of a publication but, for the sake of simplicity, no explicit mention of it will be made.

A significant part of Serre's work was given in his annual courses at the Collège de France. When we mention one of these, it will be understood to be a course held at this institution, unless otherwise stated.

1 The Beginnings

The mathematical training of J.-P. Serre can be seen as coming from two (closely related) sources. On one hand, in 1948 and just after having finished his studies at the École Normale Supérieure, he starts working at the Séminaire Cartan; this was a very active collaboration that was continued for about 6 years (giving and writing lectures in homological algebra, topology, and functions of several complex variables). On the other hand, since 1949 and for about 25 years, he works with Bourbaki.

In the fifties, Serre publishes his first papers, some of them coauthored with H. Cartan, A. Borel and G.P. Hochschild, and submits a doctoral dissertation under the supervision of Cartan. In an early publication, Borel and Serre [S17, Œ 2 (1950)] prove the impossibility of fibering an Euclidean space with compact fibers (not reduced to one point).

1.0.1. Serre's thesis was entitled *Homologie singulière des espaces fibrés. Applications* [S25, Œ 9(1951)] and was followed by several publications: [S19, Œ 4(1950)], [S20, Œ 5(1951)], [S21, Œ 6(1951)]. Its initial purpose was to compute the cohomology groups of the Eilenberg–MacLane complexes $K(\Pi, n)$ by induction on n, using the fact that the loop space of $K(\Pi, n)$ is $K(\Pi, n-1)$, combined with the loop fibration (see below). The homotopy lifting property, imposed by Serre on fiber spaces, allows him to construct a spectral sequence in singular homology, analogous to the one obtained by J. Leray (1950) in the setting of Čech theory. A dual spectral sequence also exists for cohomology. The concept of fiber space that he introduces is more general than the one that was usual at the time, allowing him to deal with loop spaces, as follows: given a pathwise-connected topological space X and a point $x \in X$, the loop space Ω at x is viewed as the fiber of a fiber space E over the base X. The elements of E are the paths of X starting at x. The crucial fact is that the map which assigns to each path its endpoint is a fibering $f : E \to X$ in the above sense. The space E is contractible and, once Leray's theory is suitably adapted, it turns to be very useful for relating the homology of Ω to that of X.

Serre's thesis contains several applications. For example, by combining Morse's theory (1938) with his own results, Serre proves that, on every compact connected Riemannian manifold, there exist infinitely many geodesics connecting any two distinct points. But, undoubtedly, the most remarkable application is the one concerning the computation of the homotopy groups of spheres, $\pi_i(S_n)$.

1.1. Homotopy Groups of Spheres. Earlier studies by H. Hopf, H. Freudenthal (1938) and others had determined the groups $\pi_i(S_n)$ for $i < n + 2$. L. Pontrjagin and G.W. Whitehead (1950) had computed the groups $\pi_{n+2}(S_n)$ and H. Hopf had proved that the group $\pi_{2n-1}(S_n)$, for n even, has $\mathbf{Z}$ as a quotient (and hence is infinite). Thanks also to Freudenthal's suspension theorem, it was known that the group $\pi_{n+k}(S_n)$ depends only on k if $n > k + 1$. However, it was not even known that the $\pi_i(S_n)$ are finitely generated groups. Serre shows that they are; what is more, he shows that the groups $\pi_i(S_n)$, for $i > n$, are *finite*, except for $\pi_{2n-1}(S_n)$ when n is even, which is the direct sum of $\mathbf{Z}$ and a finite group. Given a prime p, he also shows that the p-primary component of $\pi_i(S_n)$ is zero if $i < n + 2p - 3$ and $n \geqslant 3$; and that the p-primary component of $\pi_{n+2p-3}(S_n)$ is cyclic (he proved later that it has order p).

1.1.1. The study of the homotopy groups was pursued by Serre for about two years; the results were published in several papers: [S31, Œ 12(1952)], [S32, Œ 13(1952)], [S48, Œ 18(1953)], [S43, Œ 19(1953)], [S40, Œ 22(1953)]. He also wrote two *Comptes rendus* notes, with Cartan, on the technique of "killing homotopy groups": [S29, Œ 10(1952)], [S30, Œ 11(1952)]; and two papers with Borel on the use of Steenrod operations [S26, Œ 8(1951)], [S44 (1951)], a consequence being that the only spheres which have an almost complex structure are S_0, S_2 and S_6 (whether S_6 has a complex structure or not is still a very interesting open question, despite several attempts to prove the opposite).

1.1.2. Soon after his thesis, Serre was invited to Princeton. During his stay (January–February 1952), he realized that some kind of "localization process" is possible in the computation of homotopy groups. More precisely, the paper [S48, Œ 18(1953)] introduces a "mod $\mathcal{C}$" terminology in which a class of objects $\mathcal{C}$ is treated as "zero", as is done in arithmetic mod p. For instance, he proves that the groups $\pi_i(S_n)$, n even, are $\mathcal{C}$-isomorphic to the direct sum of $\pi_{i-1}(S_{n-1})$ and $\pi_i(S_{2n-1})$, where $\mathcal{C}$ denotes the class of the finite 2-groups.

The paper [S48, Œ 18(1953)] also shows that every connected compact Lie group is homotopically equivalent to a product of spheres, modulo certain exceptional prime numbers; for classical Lie groups they are those which are $\leqslant h$ where h is the Coxeter number. In the paper [S43, Œ 19(1953)], Serre determines the mod 2 cohomology algebra of an Eilenberg–MacLane complex $\mathrm{K}(\Pi; q)$, in the case where the abelian group Π is finitely generated. For this he combines results from both Borel's thesis and his own. He also determines the asymptotic behaviour of the Poincaré series of that algebra (by analytic arguments, similar to those used in the theory of partitions), and deduces that, for any given $n > 1$, there are infinitely many i's such that $\pi_i(S_n)$ has even order.

1.1.3. In the same paper, he computes the groups $\pi_{n+i}(S_n)$ for $i \leqslant 4$; and in [S32, Œ 13(1952)] and [S40, Œ 22(1953)], he goes up to $i \leqslant 8$. (These groups are now known for larger values of i, but there is very little information on their asymptotic behaviour for $i \to \infty$.)

1.2. Hochschild–Serre Spectral Sequence. The first study of the cohomology of group extensions was R. Lyndon's thesis (1948). Serre [S18, Œ 3(1950)] and Hochschild–Serre [S46, Œ 15(1953)] go further. Given a discrete group G, a normal subgroup K of G, and a G-module A, they construct a spectral sequence

$$H(G/K, H(K, A)) \Rightarrow H(G, A).$$

If $H^r(K, A) = 0$, for $0 < r < q$, the spectral sequence gives rise to the exact sequence

$$0 \to H^q(G/K, A^K) \to H^q(G, A) \to H^q(K, A)^{G/K} \to H^{q+1}(G/K, A^K)$$
$$\to H^{q+1}(G, A).$$

This sequence became a key ingredient in many proofs. Similar results hold for Lie algebras, as shown in [S47, Œ 16(1953)].

1.3. Sheaf Cohomology of Complex Manifolds. In his seminar at the École Normale Supérieure, Cartan showed in 1952–1953 that earlier results of K. Oka and himself can be reinterpreted and generalized in the setting of Stein manifolds by using analytic coherent sheaves and their cohomology; he thus obtained his well-known "Theorems A and B". In [S42, Œ 23(1953)] (see also the letters to Cartan reproduced in [S231 (1991)]), Serre gives several applications of Cartan's theorems; he shows for instance that the Betti numbers of a Stein manifold of complex dimension n can be computed à la de Rham, using holomorphic differential forms; in particular, they vanish in dimension $> n$. But Serre soon became more interested in compact complex manifolds, and especially in algebraic ones. A first step was the theorem (obtained in collaboration with Cartan, see [S41, Œ 24(1953)]) that the cohomology groups $H^q(X, \mathcal{F})$, associated to a compact complex manifold X and with values in an analytic coherent sheaf $\mathcal{F}$, are finite dimensional vector spaces; the proof is based on a result due to L. Schwartz on completely continuous maps between Fréchet spaces. This finiteness result played an essential role in "GAGA", see Sect. 2.2.

1.3.1. In a paper dedicated to H. Hopf, Serre [S58, Œ 28(1955)] proves a "duality theorem" in the setting of complex manifolds. The proof is based on Schwartz's theory of distributions (a distribution can be viewed either as a generalized function, or as a linear form on smooth functions; hence, distribution theory has a built-in self-duality).

1.3.2. Previously, in a letter [Œ 20(1953)] addressed to Borel, Serre had conjectured a generalization of the Riemann–Roch theorem to varieties of higher dimension. This generalization was soon proved by F. Hirzebruch in his well-known *Habilitationsschrift* and presented by Serre at the Séminaire Bourbaki. The more general version of the Riemann–Roch theorem, due to Grothendieck (1957), was the topic of a Princeton seminar by Borel and Serre. A detailed account appeared in [S74 (1958)], a paper included in [A. Borel. *Œuvres*, *Collected Papers*, no. 44]), and which was for many years the only reference on this topic.

1.4. The Amsterdam Congress. At the International Congress of Mathematicians, held in Amsterdam in 1954, K. Kodaira and J.-P. Serre were awarded the Fields Medals. With respect to Serre, the committee acknowledged the new insights he had provided in topology and algebraic geometry. At not quite 28, Serre became the youngest mathematician to receive the distinction, a record that still stands today. In his presentation of the Fields medalists, H. Weyl (perhaps a little worried by Serre's youth) recommended the two laureates to "carry on as you began!". In the following sections we shall see just how far Serre followed Weil's advice.

In his address at the International Congress [S63, Œ 27(1956)], Serre describes the extension of sheaf theory to algebraic varieties defined over a field of any characteristic (see "FAC", Sect. 2.1). One of the highlights is the algebraic analogue of the analytic "duality theorem" mentioned above (it was soon vastly generalized by Grothendieck). He also mentions the following problem: If X is a non-singular projective variety, is it true that the formula

$$B_n = \sum_{p+q=n} \dim H^q(X, \Omega^p)$$

yields the n-th Betti numbers of X occurring in Weil conjectures? This is so for most varieties, but there are counterexamples due to J. Igusa (1955).

2 Sheaf Cohomology

The paper FAC (1955) by Serre, the paper *Tôhoku* (1957) by Grothendieck, and the book by R. Godement on sheaf theory (1964) were publications that did the most to stimulate the emergence of a new methodology in topology and abstract algebraic geometry. This new approach emerged in the next twenty years through the momentous work [EGA (1960–1964)], [SGA (1968; 1971–1977)] accomplished by Grothendieck and his collaborators.

2.1. FAC. In his foundational paper entitled *Faisceaux algébriques cohérents* [S56, Œ 29(1955)], known as FAC, Serre introduces coherent sheaves in the setting of algebraic varieties over an algebraically closed field k of arbitrary characteristic. There are three chapters in FAC.

Chapter I is devoted to coherent sheaves and general sheaf theory. Chapter II starts with a sheaf-style definition of what an "algebraic variety" is (with the restriction that its local rings are reduced), and then shows that the theory of affine algebraic varieties is similar to Cartan's theory of Stein manifolds: the higher cohomology groups of a coherent sheaf are zero.

Chapter III is devoted to projective varieties. The cohomology groups of coherent sheaves are usually non-zero (but they are finite-dimensional); it is shown that they can be computed algebraically, using the "Ext" functors which had just been defined by Cartan–Eilenberg (this was their first application to algebraic geometry—there would be many others ...).

2.2. GAGA. In 1956, Serre was appointed *Professeur* at the Collège de France, in the chair *Algèbre et Géométrie*. The same year, he published the paper *Géométrie algébrique et géométrie analytique* [S57, Œ 32(1955–1956)], usually known as GAGA, in which he compares the algebraic and the analytic aspects of the complex projective varieties. The main result is the following:

> Assume that X is a projective variety over $\mathbf{C}$, and let X^h be the complex analytic space associated to X. Then the natural functor "algebraic coherent sheaves on X" $\to$ "analytic coherent sheaves on X^h" is an equivalence of categories, which preserves cohomology.

As applications, we mention the invariance of the Betti numbers under automorphisms of the complex field $\mathbf{C}$, when X is non-singular, as well as the comparison of principal algebraic fiber bundles of base X and principal analytic fiber bundles of base X^h with the same structural group G.

GAGA contains an appendix introducing the notion of flatness, and applying it to compare the algebraic and the analytic local rings of X and X^h at a given point. Flatness was to play an important role in Grothendieck's later work.

2.2.1. Let X be a normal analytic space and S a closed analytic subset of X with $\operatorname{codim}(S) \geqslant 2$ at every point. In [S120, Œ 68(1966)], Serre studies the extendibility of coherent analytic sheaves $\mathcal{F}$ on $X - S$. He shows that it is equivalent to the coherence of the direct image $i_*(\mathcal{F})$, where $i : X - S \to X$ denotes the inclusion. When X is projective, this implies that the extendible sheaves are the same as the algebraic ones.

2.3. Cohomology of Algebraic Varieties. The paper [S68, Œ 35(1957)] gives a cohomological characterization of affine varieties, similar to that of Stein manifolds (cf. [S42, Œ 23(1953)]).

2.3.1. In his lecture [S76, Œ 38(1958)] at the International Symposium on Algebraic Topology held in Mexico City, Serre associates to an algebraic variety X, defined over an algebraically closed field k of characteristic $p > 0$, its cohomology groups $H^i(X, \mathcal{W})$ with values in a sheaf of Witt vectors $\mathcal{W}$. Although this did not provide suitable Betti numbers, the paper contains many ideas that paved the way for the birth of crystalline and p-adic cohomology. We stress the treatment given in this work to the Frobenius endomorphism F, as a semilinear endomorphism of $H^1(X, \mathcal{O})$, when X is a non-singular projective curve. The space $H^1(X, \mathcal{O})$ may be identified with a space of classes of repartitions (or "adèles") over the function field of X and its Frobenius endomorphism gives the Hasse–Witt matrix of X. By using Cartier's operator on differential forms, Serre proves that $H^1(X, \mathcal{W})$ is a free module of rank $2g - s$ over the Witt ring $W(k)$, where g denotes the genus of the curve and p^s is the number of divisor classes of X killed by p.

2.3.2. The above results were completed in the paper [S75, Œ 40(1958)], dedicated to E. Artin. Given an abelian variety A, Serre shows that the cohomology algebra $H^*(A, \mathcal{O})$ is the exterior algebra of the vector space $H^1(A, \mathcal{O})$, as in the classical case. He also shows that the Bockstein's operations are zero; that is, A has no cohomological "torsion". Moreover, he gives an example of an abelian variety for which $H^2(A, \mathcal{W})$ is not a finitely generated module over the Witt vectors, thus contradicting an "imprudent conjecture" (sic) he had made in [S76, Œ 38(1958)].

2.3.3. In [S76, Œ 38(1958)], the influence of André Weil is clear (as it is in many of Serre's other papers). The search for a good cohomology for varieties defined over finite fields was motivated by the Weil conjectures (1948) on the zeta function of these varieties. As is well known, such a cohomology was developed a few years later by Grothendieck, using étale topology.

2.3.4. In [S89, Œ 45(1960)], Serre shows that Weil's conjectures could be proved easily if (a big "if") some basic properties of the cohomology of complex Kähler varieties could be extended to projective varieties over a finite field. This was the starting point for Grothendieck's formulation of the so-called "standard conjectures" on motives, which are still unproved today.

2.3.5. In [S94, Œ 50(1961)], Serre constructs a non-singular projective variety in characteristic $p > 0$ which cannot be lifted to characteristic zero. He recently (2005) improved this result by showing that, if the variety can be lifted (as a flat scheme) to a local ring A, then $p \cdot A = 0$. The basic idea consists in transposing the problem to the context of finite groups.

2.3.6. In his lecture [S103, Œ 56(1963)] at the International Congress of Mathematicians held in Stockholm in 1962, Serre offered a summary of scheme theory. After revising Grothendieck's notion of Grassmannian, Hilbert scheme, Picard scheme, and moduli scheme (for curves of a given genus), he goes on to schemes over complete noetherian local rings, where he mentions the very interesting and, in those days, recent results of Néron. (Although the language of schemes would become usual in Serre's texts, it is worth saying that he has never overused it.)

3 Lie Groups and Lie Algebras

Serre's interest in Lie theory was already apparent in his complementary thesis [S28, Œ 14(1952)], which contains a presentation of the results on Hilbert's fifth problem up to 1951 (i.e. just before it was solved by A.M. Gleason, D. Montgomery and L. Zippin).

3.0.1. In 1953, Borel and Serre began to be interested in the finite subgroups of compact Lie groups—a topic to which they would return several times in later years, see e.g. [S250, Œ 167(1996)] and [S260, Œ 174(1999)]. In [S45 (1953)], reproduced in [A. Borel. *Œuvres*, *Collected Papers*, no. 24], they prove that every supersolvable finite subgroup of a compact Lie group G is contained in the normalizer N of a maximal torus T of G; this is a generalization of a theorem of Blichfeldt relative to $G = \mathbf{U}_n$. In particular, the determination of the abelian subgroups of G is reduced to that of the abelian subgroups of N. Borel and Serre were especially interested in the p-elementary abelian subgroups of G, for p a prime. They defined the p-rank ℓ_p of G as the largest integer n such that G contains such a subgroup with order p^n; the theorem above shows that $\ell \leqslant \ell_p(G) \leqslant \ell + \ell_p(W)$, where ℓ is the rank of G and $\ell_p(W)$ the p-rank of its Weyl group $W = N/T$. They show that, if G is connected and its p-rank is greater than its rank, then G has homological p-torsion. As a corollary, the compact Lie groups of type G_2, F_4, E_7 and E_8 have homological 2-torsion. The proof uses results on the cohomology algebra modulo p of the classifying space B_G of G, which had been studied in Borel's thesis.

3.0.2. Serre's book *Lie Algebras and Lie Groups* [S110, LALG(1965)] was based on a course at Harvard. As indicated by its title, it consists of two parts; the first one gives the general theory of Lie algebras in characteristic zero, including the standard theorems of Lie, Engel, Cartan and Whitehead (but not including root systems). The second one is about analytic manifolds over a complete field k, either real, complex or ultrametric. It is in this context that Serre gives the standard Lie dictionary

$$\text{Lie groups} \to \text{Lie algebras},$$

assuming that k has characteristic zero. His interest in the p-adic case arose when he realized around 1962 that the rather mysterious Galois groups associated to the Tate modules of abelian varieties (see Sect. 13) are p-adic Lie groups, so that their Lie algebras are accessible to the general Lie theory. This elementary remark opened up many possibilities, since it is much easier to classify Lie algebras than profinite groups.

3.0.3. The booklet *Algèbres de Lie semi-simples complexes* [S119, ALSC(1966)] reproduces a series of lectures given in Algiers in 1965. It gives a concise introduction (mostly with proofs) to complex semisimple Lie algebras, and thus supplements *Lie Algebras and Lie Groups*. The main chapters are those on Cartan subalgebras, representation theory for $\mathfrak{sl}_2$, root systems and their Weyl groups, structure theorems for semisimple Lie algebras, linear representations of semisimple Lie algebras, Weil's character formula (without proof), and the dictionary between compact Lie groups and reductive algebraic groups over **C** (without proof, but see *Gèbres* [S239, Œ 160(1993)]). The book also gives a presentation of semisimple Lie algebras by generators and relations (including the so-called "Serre relations" which, as he says, should be called "Chevalley relations" because of their earlier use by Chevalley).

4 Local Algebra

Serre's work in FAC and GAGA made him introduce homological methods in local algebra, such as *flatness* and the characterization of regular local rings as the only noetherian local rings of finite homological dimension (completing an earlier result of A. Auslander and D. Buchsbaum (1956), cf. [S64, Œ 33(1956)]).

4.0.1. The general theory of local rings was the subject of the lecture course [S79, Œ 42(1958)], which was later published in book form *Algèbre Locale. Multiplicités* [S111, ALM(1965)]. Its topics include the general theory of noetherian modules and their primary decomposition, Hilbert polynomials, integral extensions, Krull–Samuel dimension theory, the Koszul complex, Cohen–Macaulay modules, and the homological characterization of regular local rings mentioned above. The book culminates with the celebrated "Tor formula" which gives a homological definition for intersection multiplicities in algebraic geometry in terms of Euler–Poincaré characteristics. This led Serre to several conjectures on regular local rings of mixed characteristic; most of them (but not all) were later proved by P. Roberts (1985), H. Gillet–C. Soulé (1985) and O. Gabber. The book had a profound influence on a whole generation of algebraists.

5 Projective Modules

Given an algebraic vector bundle E over an algebraic variety V, let $\mathcal{S}(E)$ denote its sheaf of sections. As pointed out in FAC, one gets in this way an equivalence between vector bundles and locally free coherent $\mathcal{O}$-sheaves. When V is affine, with coordinate ring $A = \Gamma(V, \mathcal{O})$, this may be viewed as a correspondence between vector bundles and finitely generated projective A-modules; under this correspondence trivial bundles correspond to free modules.

5.0.1. The above considerations apply when V is the affine n-space over a field k, in which case A is the polynomial ring $k[X_1, \ldots, X_n]$; Serre mentions in FAC that he "does not know of any finitely generated projective $k[X_1, \ldots, X_n]$-module which is not free". This gave rise to the so-called *Serre conjecture*, although it had been stated as a "problem" and not as a "conjecture". Much work was done on it (see e.g. the book by T.Y. Lam called *Serre's Conjecture* in its 1977 edition and *Serre's Problem* in its 2006 one). The case $n = 2$ was solved by C.S. Seshadri (1979); see Serre's report on it in Séminaire Dubreil–Pisot [Œ 48(1960/61)]; this report also gives an interesting relation between the problem for $n = 3$ and curves in affine 3-space which are complete intersections.

Twenty years after the publication of FAC, and after partial results had been obtained by several authors (especially in dimension 3), D. Quillen (1976) and A. Suslin (1976), independently and simultaneously, solved Serre's problem in any dimension.

5.0.2. In his contribution [S73, Œ 39(1958)] at the Séminaire Dubreil–Dubreil-Jacotin–Pisot, Serre applies the "projective modules = vector bundles" idea to an arbitrary commutative ring A. Guided by transversality arguments of topology, he proves the following splitting theorem:

> Assume A is commutative, noetherian, and that Spec(A) is connected. Then every finitely generated projective A-module is the direct sum of a free A-module and of a projective A-module the rank of which does not exceed the dimension of the maximal spectrum of A.

When $\dim(A) = 1$, one recovers the theorem of Steinitz–Chevalley on the structure of the torsion-free modules over Dedekind rings.

6 Algebraic Number Fields

The Séminaire Bourbaki report [S70, Œ 41(1958)] contains an exposition of Iwasawa's theory for the p-cyclotomic towers of number fields, and the p-components of their ideal class groups. The main difference with Iwasawa's papers is that the structure theorems for the so-called Γ-modules are deduced from general statements on regular local rings of dimension 2; this viewpoint has now become the standard approach to such questions.

6.0.1. *Cours d'Arithmétique* [S146, CA(1970)] arose as a product of two lecture courses taught in 1962 and 1964 at the École Normale Supérieure. The book (which in its first edition had the format 11 cm $\times$ 18 cm and cost only 12 francs) has been frequently translated and reprinted, and has been the most accessible introduction to certain chapters of number theory for many years. The first part, which is purely algebraic, gives the classification of quadratic forms over $\mathbf{Q}$. We find there equations over finite fields, two proofs of the quadratic reciprocity law, an introduction to p-adic numbers, and properties of the Hilbert symbol. The quadratic forms are studied over $\mathbf{Q}_p$, over $\mathbf{Q}$, as well as over $\mathbf{Z}$ (in the case of discriminant ± 1). The second part of the book uses analytic methods. It contains a chapter on L-functions, culminating in the standard proof of Dirichlet's theorem on primes in arithmetic progressions, and a chapter on modular forms of level 1, together with their relations with elliptic curves, Eisenstein series, Hecke operators, theta functions and

Ramanujan's τ function. In 1995, Serre was awarded the Leroy P. Steele Prize for Mathematical Exposition for this delightful text.

6.0.2. The notion of a *p-adic modular form* was introduced by Serre in the paper [S158, Œ 97(1973)], which is dedicated to C.L. Siegel. Such a form is defined as a limit of modular forms in the usual sense. By using them, together with previous results on modular forms mod p due to Swinnerton-Dyer and himself, he constructs the *p-adic zeta function* of a totally real algebraic number field K. This function interpolates p-adically the values at the negative integers of the Dedekind zeta function $\zeta_K(s)$ (after removal of its p-factors); these numbers were already known to be rational, thanks to a theorem of Siegel (1937). Serre's results generalize the one obtained by Kubota–Leopoldt in the sixties when K is abelian over $\mathbf{Q}$. They were completed later by D. Barsky (1978), Pierrette Cassou-Noguès (1979) and P. Deligne–K. Ribet (1980).

7 Class Field Theory

Class field theory describes the abelian extensions of certain fields by means of what are known as reciprocity isomorphisms. Sometimes, the reciprocity isomorphisms can be made explicit by means of a symbol computation. The first historical example is the quadratic reciprocity law of Legendre and Gauss. The cohomological treatment of class field theory started with papers of G.P. Hochschild, E. Artin, J. Tate, A. Weil and T. Nakayama.

7.1. Geometric Class Field Theory. *Groupes Algébriques et Corps de Classes* [S82, GACC(1959)] was the first book that Serre published. It evolved from his first course at the Collège de France [S69, Œ 37(1957)] and its content is mainly based on earlier papers by S. Lang (1956) and M. Rosenlicht (1957).

Chapter I is a résumé of the book. Chapter II gives the main theorems on algebraic curves, including Riemann–Roch and the duality theorem (with proofs). Chapters III–IV are devoted to a theorem of Rosenlicht stating that every rational function $f : X \to G$, from a non-singular irreducible projective curve X to a commutative algebraic group G, factors through a generalized Jacobian $J_{\mathfrak{m}}$. A generalized Jacobian is a commutative algebraic group, which is an extension of an abelian variety (the usual Jacobian J) by an algebraic linear group $L_{\mathfrak{m}}$, depending on a "modulus" $\mathfrak{m}$. The groups $L_{\mathfrak{m}}$ provide the local symbols in class field theory. In Chap. V it is shown that every abelian covering of an irreducible algebraic curve is the pull-back of a separable isogeny of a generalized Jacobian. When $\mathfrak{m}$ varies, the generalized Jacobians $J_{\mathfrak{m}}$ form a projective system which is the geometric analogue of the idèle class group. Class field theory for function fields in one variable over finite fields is dealt with in Chap. VI. The reciprocity isomorphism is proved and explicit computations of norm residue symbols are made. Chapter VII contains a general cohomological treatment of extensions of commutative algebraic groups.

7.1.1. Based on the lecture course [S91, Œ 47(1960)], a theory of commutative pro-algebraic groups is developed in [S88, Œ 49(1960)]. Its application to geometric class field theory can be found in [S93, Œ 51(1961)].

Let k be an algebraically closed field. A commutative quasi-algebraic group over k is defined as a pure inseparable isogeny class of commutative algebraic groups over k. If G is such a group and is connected, then it has a unique connected linear subgroup L such that the quotient G/L is an abelian variety. As for the group L, it is the product of a torus T by a unipotent group U. The group T is a product of groups isomorphic to $\mathbf{G}_m$ and the group U has a composition series whose quotients are isomorphic to $\mathbf{G}_a$; it is isogenous to a product of truncated Witt vector groups. The isomorphism classes of the groups $\mathbf{G}_a$, $\mathbf{G}_m$, cyclic groups of prime order, and simple abelian varieties are called the elementary commutative quasi-algebraic groups. The commutative quasi-algebraic groups form an abelian category $\mathcal{Q}$. The finite commutative quasi-algebraic groups form a subcategory $\mathcal{Q}_0$ of $\mathcal{Q}$. If G is a commutative quasi-algebraic group and G^0 is its connected component, the quotient $\pi_0(G) = G/G^0$ is a finite abelian group. The category of commutative pro-algebraic groups is defined as $\mathcal{P} = \mathrm{Pro}(\mathcal{Q})$. Let $\mathcal{P}_0 = \mathrm{Pro}(\mathcal{Q}_0)$ be the subcategory of abelian profinite groups. The category $\mathcal{P}$ has projective limits and enough projective objects. Every projective object of $\mathcal{P}$ is a product of indecomposable projectives and the indecomposable projective groups coincide with the projective envelopes of the elementary commutative quasi-algebraic groups. The functor $\pi_0 : \mathcal{P} \to \mathcal{P}_0$ is right exact; its left derived functors are denoted by $G \mapsto \pi_i(G)$. One of the main results of the paper is that $\pi_i(G) = 0$ if $i > 1$. The group $\pi_1(G)$ is called the fundamental group of G. The connected and simply connected commutative pro-algebraic groups form a subcategory $\mathcal{S}$ of $\mathcal{P}$. For each object G in $\mathcal{P}$, there exists a unique group $\widetilde{G}$ in $\mathcal{S}$ and a morphism $u : \widetilde{G} \to G$, whose kernel and cokernel belong to $\mathcal{P}_0$, so that one obtains an exact sequence

$$1 \to \pi_1(G) \to \widetilde{G} \to G \to \pi_0(G) \to 1.$$

By means of the universal covering functor, the categories $\mathcal{P}/\mathcal{P}_0$ and $\mathcal{S}$ become equivalent.

After computing the homotopy groups of the elementary commutative pro-algebraic groups, it is shown in [S88, Œ 49(1960)] that every commutative pro-algebraic group has cohomological dimension $\leqslant 2$, if k has positive characteristic; and has cohomological dimension $\leqslant 1$, if k has characteristic zero.

7.1.2. The paper [S93, Œ 51(1961)] is a sequel to the one mentioned above (and is also its motivation). It deals with local class field theory in the geometric setting (in an Oberwolfach lecture, Serre once described it as *reine geometrische Klassenkörpertheorie im Kleinen*). Let K be a field which is complete with respect to a discrete valuation and suppose that its residue field k is algebraically closed. By using a construction of M. Greenberg, the group of units U_K of K may be viewed as a commutative pro-algebraic group over k, so that the fundamental group

$\pi_1(U_K)$ is well defined. The reciprocity isomorphism takes the simple form:

$$\pi_1(U_K) \xrightarrow{\sim} G_K^{\mathrm{ab}},$$

where G_K^{ab} denotes the Galois group of the maximal abelian extension of K. This isomorphism is compatible with the natural filtration of $\pi_1(U_K)$ and the filtration of G_K^{ab} given by the upper numbering of the ramification groups. Hence there is a conductor theory, related to Artin representations (see below).

7.1.3. An Artin representation a has a $\mathbf{Z}$-valued character. In the paper [S90, Œ 46(1960)], it is shown that a is rational over $\mathbf{Q}_\ell$ provided ℓ is different from the residue characteristic, but it is not always rational over $\mathbf{Q}$. The same paper conjectures the existence of a conductor theory for regular local rings of any dimension, analogous to the one in dimension 1; a few results have been obtained on this recently by K. Kato and his school, but the general case is still open.

7.1.4. An example, in the geometric case, of a separable covering of curves with a relative different whose class is not a square was given in a joint paper with A. Fröhlich and J. Tate [S100, Œ 54(1962)]. Such an example does not exist for number fields, by a well-known result of E. Hecke.

7.1.5. In [S150, Œ 92(1971)], Serre considers a Dedekind ring A of field of fractions K, a finite Galois extension L/K with Galois group G, and a real-valued virtual character χ of G. Under the assumption that either the extension L/K is tamely ramified or that χ can be expressed as the difference of two characters of real linear representations, he proves that the Artin conductor $\mathfrak{f} = \mathfrak{f}(\chi, L/K)$ is a square in the group of ideal classes of A.

7.2. Local Class Field Theory. Group cohomology and, more specifically, Galois cohomology is the subject of the lecture course [S83, Œ 44(1959)]. The content of this course can be found in *Corps Locaux* [S98, CL(1962)].

The purpose of [CL] was to provide a cohomological presentation of local class field theory, for valued fields which are complete with respect to a discrete valuation with finite residue field. In the first part, one finds the structure theorem of complete discrete valuation rings. In the second part, Hilbert's ramification theory is given, with the inclusion of the upper numbering of the ramification groups, due to J. Herbrand, and the properties of the Artin representation, a notion due to Weil in his paper *L'avenir des mathématiques* (1947). The third part of [CL] is about group cohomology. It includes the cohomological interpretation of the Brauer group $\mathrm{Br}(k)$ of a field k and class-formations à la Artin–Tate.

Local class field theory takes up the fourth part of the book. The reciprocity isomorphism is obtained from the class formation associated to the original local field; it is made explicit by means of a computation of norm residue symbols based on a theorem of B. Dwork (1958).

One also finds in [CL] the first definitions of non-abelian Galois cohomology. Given a Galois extension K/k and an algebraic group G defined over k, the elements of the set $H^1(\mathrm{Gal}(K/k), G(K))$ describe the classes of principal homogeneous G-spaces over k which have a rational point in K. Easy arguments show that $H^1(\mathrm{Gal}(K/k), G(K)) = 1$ when G is one of the following algebraic groups: additive $\mathbf{G}_a$, multiplicative $\mathbf{G}_m$, general linear $\mathbf{GL}_n$, and symplectic $\mathbf{Sp}_{2n}$.

7.2.1. Another exposition of local class field theory can be found in the lecture [S127, Œ 75(1967)]; it differs from the one given in [CL] by the use of Lubin–Tate theory of formal groups, which allows a neat proof of the "existence theorem".

7.3. A Local Mass Formula. Let K denote a local field with finite residue field k of q elements and let K_s be a separable closure of K. In [S186, Œ 115(1978)], one finds a *mass formula* for the set Σ_n of all totally ramified extensions of K of given degree n contained in K_s, namely:

$$\sum_{L\in\Sigma_n} 1/q^{c(L)} = n,$$

where $q^{c(L)}$ is the norm of the wild component of the discriminant of L/K. Although the formula could (in principle) be deduced from earlier results of Krasner, Serre proves it independently in two elegant and different ways. The first proof is derived from the volume of the set of Eisenstein polynomials. The second uses the p-adic analogue of Weil's integration formula, applied to the multiplicative group D^* of a division algebra D of center K such that $[D : K] = n^2$.

8 p-adic Analysis

Let V be an algebraic variety over a finite field k of characteristic p. One of Weil's conjectures is that the zeta function $Z_V(t)$ is a rational function of t. This was proved in 1960 by B. Dwork. His method involved writing $Z_V(t)$ as an alternating product of p-adic Fredholm determinants. This motivated Serre to study the spectral theory of completely continuous operators acting on p-adic Banach spaces [S99, Œ 55(1962)]. The paper, which is self-contained, provides an excellent introduction to p-adic analysis. Given a completely continuous endomorphism u defined on a Banach space E over a local field, the Fredholm determinant $\det(1-tu)$ is a power series in t, which has an infinite radius of convergence and thus defines an entire function of t. The Fredholm resolvent $P(t,u) = \det(1-tu)/(1-tu)$ of u is an entire function of t with values in $\mathrm{End}(E)$. Given an element $a \in K$, one shows that the endomorphism $1-au$ is invertible if and only if $\det(1-au) \neq 0$. If this is the case, then the relation $\det(1-au) = (1-au)P(a,u) = P(a,u)(1-au)$ is satisfied. If $a \in K$ is a zero of order h of the function $\det(1-tu)$, then the space E uniquely decomposes into a direct sum of two closed subspaces N, F

which are invariant under u. The endomorphism $1 - au$ is nilpotent on N and invertible on F; the dimension of N is h, just as in F. Riesz theory over $\mathbf{C}$. Serre proves that, given an exact sequence of Banach spaces and continuous linear mappings, $0 \to E_0 \xrightarrow{d_0} E_1 \to \cdots \xrightarrow{d_{n-1}} E_n \to 0$, and given completely continuous endomorphisms u_i of E_i such that $d_i \circ u_i = u_{i+1} \circ d_i$, for $0 \leqslant i < n$, then $\prod_{i=1}^{n} \det(1 - tu_i)^{(-1)^i} = 1$; this is useful for understanding some of Dwork's computations.

8.0.1. In [S113, Œ 65(1965)], the compact p-adic analytic manifolds are classified. Given a field K, locally compact for the topology defined by a discrete valuation, any compact analytic manifold X defined over K, of dimension n at each of its points, is isomorphic to a disjoint finite sum of copies of the ball A^n, where A denotes the valuation ring of K. Two sums rA^n and $r'A^n$ are isomorphic if and only if $r \equiv r' \bmod (q-1)$, where q denotes the number of elements of the residue field of A. The class of $r \bmod (q-1)$ is an invariant of the manifold; two n-manifolds with the same invariant are isomorphic.

9 Group Cohomology

By definition, a profinite group is a projective limit of finite groups; special cases are the pro-p-groups, i.e. the projective limits of finite p-groups. The most interesting examples of profinite groups are provided by the Galois groups of algebraic extensions and by compact p-adic Lie groups.

9.1. Cohomology of Profinite Groups and p-adic Lie Groups. To each profinite group $G = \varprojlim G_i$ acting in a continuous way on a discrete abelian group A, one can associate cohomology groups $H^q(G, A)$ by using continuous cochains. The main properties of the cohomology of profinite groups were obtained by Tate (and also by Grothendieck) in the early 1960s, but were not published. They are collected in the first chapter of *Cohomologie Galoisienne* [S97, CG(1962)].

In the first chapter of [CG], given a prime p and a profinite group G, the concepts of cohomological p-dimension, denoted by $\mathrm{cd}_p(G)$, and cohomological dimension, denoted by $\mathrm{cd}(G)$, are defined. Some pro-p-groups admit a duality theory; they are called Poincaré pro-p-groups. Those of cohomological dimension 2 are the "Demushkin groups". They are especially interesting, since they can be described by one explicit relation (Demushkin, Serre, Labute); they appear as Galois groups of the maximal pro-p-extension of p-adic fields, cf. [S104, Œ 58(1963)] and [CG].

Chapters II and III are devoted to the study of Galois cohomology in the commutative and the non-commutative cases (most of the results of Chap. II were due to Tate, and an important part of those of Chap. III were due to Borel–Serre).

9.1.1. The Bourbaki report [S105, Œ 60(1964)] summarizes M. Lazard's seminal paper (1964) on p-adic Lie groups. One of Lazard's main results is that a profinite group is an analytic p-adic group if and only if it has an open subgroup H which is a pro-p-group and is such that $(H, H) \subset H^p$, if $p \neq 2$; or $(H, H) \subset H^4$, if $p = 2$. If G is a compact p-adic Lie group such that $\mathrm{cd}(G) = n < \infty$, then G is a Poincaré pro-p-group of dimension n and the character $\chi(x) = \det \mathrm{Ad}(x)$ is the dualizing character of G. Here $\mathrm{Ad}(x)$ denotes the adjoint automorphism of $\mathrm{Lie}(G)$ defined by x. The group G has finite cohomological dimension if and only if it is torsion-free; the proof combines Lazard's results with the theorem of Serre mentioned below.

9.1.2. The paper [S114, Œ 66(1965)] proves that, if G is a profinite group, p-torsion-free, then for every open subgroup U of G we have the equality $\mathrm{cd}_p(U) = \mathrm{cd}_p(G)$ between their respective cohomological p-dimensions. The proof is rather intricate. In it, Serre makes use of Steenrod powers, a tool which he had acquired during his topological days (cf. [S44 (1953)]). As a corollary, every torsion-free pro-p-group which contains a free open subgroup is free. Serre asked whether the discrete analogue of this statement is true, i.e. whether every torsion-free group G which contains a free subgroup of finite index is free. This was proved a few years later by J. Stallings (1968) and R. Swan (1969).

9.1.3. More than thirty years later, Serre dedicated [S255, Œ 173(1998)] to John Tate. The paper deals with the Euler characteristic of profinite groups. Given a profinite group G of finite cohomological p-dimension and a discrete G-module A which is a vector space of finite dimension over the finite field $\mathbf{F}_p$, the Euler characteristic

$$e(G, A) = \sum(-1)^i \dim H^i(G, A)$$

is defined under the assumption that $\dim H^i(G, A) < \infty$, for all i.

Let G_{reg} be the subset of G made up by the regular elements. Serre proves that there exists a distribution μ_G over G_{reg} with values in $\mathbf{Q}_p$ such that $e(G, A) = \langle \varphi_A, \mu_G \rangle$, where $\varphi_A : G_{\mathrm{reg}} \to \mathbf{Z}_p$ denotes the Brauer character of the G-module A. This distribution can be described explicitly in several cases, e.g. when G is a p-torsion-free p-adic Lie group, thanks to Lazard's theory.

9.2. Galois Cohomology. Let $G = \mathrm{Gal}(K/k)$ be the Galois group of a field extension and suppose that A is a discrete G-module. The abelian Galois cohomology groups $H^q(\mathrm{Gal}(K/k), A)$ are usually denoted by $H^q(K/k, A)$, or simply by $H^q(k, A)$ when $K = k_s$ is a separable closure of k.

Abelian Galois cohomology, with special emphasis on the results of Tate, was the content of the course [S104, Œ 59(1963)] and of the second chapter of [CG], while the third chapter of [CG] is about non abelian cohomology. After thirty years, Serre returned to both topics in a series of three courses [S234, Œ 153(1991)], [S236, Œ 156(1992)], [S247, Œ 165(1994)].

9.3. Galois Cohomology of Linear Algebraic Groups. In his lecture delivered at Brussels in the Colloquium on Algebraic Groups [S96, Œ 53(1962)], Serre presented two conjectures on the cohomology of linear algebraic groups, known as Conjecture I (CI) and Conjecture II (CII).

Given an algebraic group G defined over a field k, we may consider the cohomology group $H^0(k,G)=G(k)$, and the cohomology set $H^1(k,G)$ = isomorphism classes of G-k-torsors.

In what follows, we will suppose that the ground field k is perfect, and we will denote by cd(k) the cohomological dimension of Gal(k_s/k). The above conjectures state:

(CI) If $\mathrm{cd}(k)\leqslant 1$ and G is a connected linear group, then $H^1(k,G)=0$.
(CII) If $\mathrm{cd}(k)\leqslant 2$ and G is a semisimple, simply connected linear group, then $H^1(k,G)=0$.

At the time, the truth of Conjecture I was only known in the following cases:

- when k is a finite field (S. Lang);
- when k is of characteristic zero and has property "C_1" (T. Springer);
- for G solvable, connected and linear;
- for G a classical semisimple group.

Conjecture I was proved a few years later in a beautiful paper by R. Steinberg (1965), which Serre included in the English translation of [CG].

M. Kneser (1965) proved Conjecture II when k is a p-adic field, and G. Harder (1965) did the same when k is a totally imaginary algebraic number field and G does not have any factor of type E_8; this restriction was removed more than 20 years later by V.I. Chernousov (1989). More generally, for any algebraic number field k, and any semisimple, simply connected linear algebraic group G, the natural mapping $H^1(k,G)\to\prod_{v\mathrm{real}}H^1(k_v,G)$ is bijective (Hasse's principle), in agreement with a conjecture of Kneser. A detailed presentation of this fact can be found in a book by V. Platonov and A. Rapinchuk (1991).

9.3.1. The Galois cohomology of semisimple linear groups was taken up again by Serre in the course [S234, Œ 153(1991)]. One of his objectives was to discuss the "cohomological invariants" of $H^1(k,G)$, i.e. (see below) the relations which connect the non-abelian cohomology set $H^1(k,G)$ and certain Galois cohomology groups $H^i(k,C)$, where C is commutative (e.g. $C=\mathbf{Z}/2\mathbf{Z}$).

More precisely, let us consider a smooth linear algebraic group G, defined over a field k_0, an integer $i\geqslant 0$, and a finite Galois module C over k_0 whose order is coprime to the characteristic.

By definition, a cohomological invariant of type $H^i(-,C)$ is a morphism of the functor $k\mapsto H^1(k,G)$ into the functor $k\mapsto H^i(k,C)$, defined over the category of field extensions k of k_0. Suppose that the characteristic of k is not 2. Then, examples of cohomological invariants are provided by

- the Stiefel–Whitney classes $w_i: H^1(k,\mathbf{O}(q))\to H^i(k,\mathbf{Z}/2\mathbf{Z})$;
- Arason's invariant $a: H^1(k,\mathbf{Spin}(q))\to H^3(k,\mathbf{Z}/2\mathbf{Z})$;
- Merkurjev–Suslin's invariant $ms: H^1(k,\mathbf{SL}_D)\to H^3(k,\mu_n^{\otimes 2})$;
- Rost's invariants.

9.3.2. The presentation of recent work on Galois cohomology and the formulation of some open problems was the purpose of the exposé [S246, Œ 166(1994)] at the Séminaire Bourbaki. To every connected semisimple group G whose root system over k is irreducible, one associates a set of prime numbers $S(G)$ which plays a special role in the study of the cohomology set $H^1(k, G)$. For example, all the divisors of the order of the centre of the universal covering $\widetilde{G}$ of G are included in $S(G)$. Tits' theorem (1992) proves that, given a class $x \in H^1(k, G)$, there exists an extension k_x/k of $S(G)$-primary degree that kills x (that is to say, such that x maps to zero in $H^1(k_x, G)$). Serre asks whether it is true that, given finite extensions k_i/k whose degrees are coprime to $S(G)$, the mapping $H^1(k, G) \to \prod H^1(k_i, G)$ is injective (assuming that G is connected). In this lecture, he also gives extensions and variants of Conjectures I and II which deal with an imperfect ground field or for which one merely assumes that $\mathrm{cd}_p(G) \leqslant 1$ for every $p \in S(G)$. He gives a list of cases in which Conjecture II has been proved, namely:

- groups of type $\mathbf{SL}_D$ associated to elements of norm 1 of a central simple k-algebra D, of rank n^2, by A.S. Merkurjev and A. Suslin (1983, 1985);
- Spin groups (in particular, all those of type B_n), by A.S. Merkurjev;
- classical groups (except those of triality type D_4), by Eva Bayer and Raman Parimala (1995);
- groups of type G_2 and F_4.

In conclusion, Conjecture II remains open for the types E_6, E_7, E_8 and triality type D_4.

9.4. Self-dual Normal Basis. E. Bayer-Fluckiger and H.W. Lenstra (1990) defined the notion of a "self-dual normal base" in a G-Galois algebra L/K, and proved the existence of such a base when G has odd order. When G has even order, existence criteria were given by E. Bayer and Serre [S244, Œ 163(1994)] in the special case where the 2-Sylow subgroups of G are elementary abelian: if 2^d is the order of such a Sylow group, they associate to L/K a d-Pfister form q_L and show that a self-dual normal base exists if and only if q_L is hyperbolic. (Thanks to Voevodsky's proof of Milnor's conjecture, this criterion can also be stated as the vanishing of a specific element of $H^d(K, \mathbf{Z}/2\mathbf{Z})$.)

9.4.1. In the Oberwolfach announcement [S275 (2005)], Serre gives a criterion for the existence of a self-dual normal base for a finite Galois extension L/K of a field K of characteristic 2. He proves that such a base exists if and only if the Galois group of L/K is generated by squares and by elements of order 2. Note that the criterion does not depend on K, nor on the extension L, but only on the structure of G. The proof uses unpublished results of his own on the cohomology of unitary groups in characteristic 2.

9.5. Essential Dimension. Let G be a simple algebraic group of adjoint type defined over a field k. The essential dimension of G is, by definition, the essential dimension of the functor of G-torsors $F \to H^1(F, G)$, which is defined over the category of field extensions F of k (in loose terms, it is the minimal number of "parameters" one needs in order to write a generic G-torsor). Here $H^1(F, G)$ denotes the non-abelian Galois cohomology set of G. In [S277 (2006)], Serre and V. Chernousov give a lower bound for the essential dimension at a prime $p = 2$, $\mathrm{ed}(G, 2)$, and for the essential dimension $\mathrm{ed}(G)$ of G. It is proved in the paper that $\mathrm{ed}(G, 2) \geqslant r + 1$ and, thus, $\mathrm{ed}(G) \geqslant r + 1$, with $r = \mathrm{rank} G$. Lower bounds for $\mathrm{ed}(G, p)$ had been obtained earlier by Z. Reichstein and B. Youssin (2000). In their proof, these authors made use of resolution of singularities, so that their results were only valid for fields k of characteristic zero (however a recent paper of P. Gille and Z. Reichstein has removed this restriction). The proof of Chernousov–Serre for $p = 2$ is valid in every characteristic different from 2. It makes use of the existence of suitable orthogonal representations of G attached to quadratic forms. The quadratic forms involved turn out to be the normalized Killing form. (We should point out that the bound they obtain has now been superseded—especially for the Spin groups.)

10 Discrete Subgroups

The study of discrete subgroups of Lie groups goes back to F. Klein and H. Poincaré.

Let us consider a global field k and a finite set S of places of k containing the set S_∞ of all the archimedean places. Let O be the ring of S-integers of k and let us denote by $\mathbf{A}_k$ and $\mathbf{A}_k^S$ the ring of adeles and of S-adeles of k, respectively. We write $\mathbf{A}_k^f$ for the ring of finite adeles of k, obtained by taking $S = S_\infty$. Given a linear algebraic group G defined over k, we shall consider a fixed faithful representation $G \to \mathbf{GL}_n$. Let $\Gamma := G(k) \cap \mathbf{GL}_n(O)$.

In $G(k)$ we may distinguish two types of subgroup, namely, the S-arithmetic subgroups and the S-congruence subgroups. A subgroup Γ' of $G(k)$ is said to be S-arithmetic if $\Gamma \cap \Gamma'$ is of finite index in both Γ and Γ'.

Let $\mathfrak{q} \subset O$ be an ideal and $\mathrm{GL}_n(O, \mathfrak{q}) := \ker(\mathbf{GL}_n(O) \to \mathbf{GL}_n(O/\mathfrak{q}))$. We define $\Gamma_\mathfrak{q} = \Gamma \cap \mathrm{GL}_n(O, \mathfrak{q})$. A subgroup Γ'' of $G(k)$ is said to be an S-congruence subgroup if it is S-arithmetic and it contains a subgroup $\Gamma_\mathfrak{q}$, for some non-zero ideal $\mathfrak{q}$. The "S-congruence subgroup problem" is the question: is every S-arithmetic subgroup an S-congruence subgroup? If $S = S_\infty$, one just refers to the "congruence subgroup problem".

Since an S-congruence subgroup is S-arithmetic, there is a homomorphism of topological groups $\pi : \widehat{G(k)} \to \overline{G(k)}$, where $\widehat{G(k)}$ denotes the completion of $G(k)$ in the topology defined by the S-congruence subgroups and $\overline{G(k)}$, the completion in that of the S-arithmetic subgroups. The group $\overline{G(k)}$ can be identified with the closure of $G(k)$ in $G(\mathbf{A}_k^f)$. Let $C^S(G)$ denote the kernel of π. The group $C^S(G)$, which coincides with the $\ker(\pi)$ restricted to $\widehat{\Gamma}$, is profinite and π is an epimorphism.

The S-congruence subgroup problem has a positive answer if and only if the "congruence kernel" $C^S(G)$ is trivial, i.e. π is an isomorphism. It is so when G is a torus (Chevalley, 1951), or is unipotent. When G is semisimple and not simply connected, the problem has a negative answer. Hence the most interesting case is when G is semisimple and simply connected.

10.1. Congruence Subgroups. Recall that a semisimple group over k is said to be split (or to be a "Chevalley group") if it has a maximal torus which splits over k.

Split groups provide a suitable framework for the study of the congruence subgroup problem. In [S126, Œ 74(1967), Œ 103(1975)], H. Bass, J. Milnor and Serre formulate the S-congruence subgroups conjecture precisely in the following form: if G is split, of rank $\geqslant 2$, simply connected and quasi simple, then the group extension $1 \to C^S(G(k)) \to \widehat{G(k)} \to \overline{G(k)} \to 1$ is central and, moreover, $C^S(G)$ is trivial unless k is totally imaginary; in the latter case $C^S(G) = \mu(k)$ is the finite subgroup consisting of the roots of unity of k^*.

10.1.1. Previously, Bass, Lazard and Serre [S108, Œ 61(1964)] had proved the congruence subgroup conjecture for $\mathbf{SL}_n(\mathbf{Z})$, $n \geqslant 3$, and $\mathbf{Sp}_{2n}(\mathbf{Z})$ for $n \geqslant 2$: every arithmetic subgroup is a congruence subgroup. The same result had been obtained independently by J. Mennicke. Bass–Lazard–Serre's proof is by induction on $n \geqslant 3$. It relies on a computation of the cohomology of the profinite groups $\mathbf{SL}_2(\mathbf{Z}_p)$, $\mathbf{Sp}_{2n}(\mathbf{Z}_p)$ with coefficients in $\mathbf{Q}/\mathbf{Z}$ and in $\mathbf{Q}_p/\mathbf{Z}_p$; this computation is made possible by Lazard's results (see above) on the cohomology of p-adic Lie groups.

10.1.2. In [S126, Œ 74(1967), Œ 103(1975)], Bass, Milnor and Serre prove the congruence subgroup conjecture when k is an algebraic number field, $G = \mathbf{SL}_n$ for $n \geqslant 3$, and $G = \mathbf{Sp}_{2n}$ for $n \geqslant 2$. In order to do this, they determine the corresponding universal Mennicke symbols associated to these groups and to the ring of integers of a totally imaginary algebraic number field k.

10.1.3. The solution of the congruence subgroup problem in the case where G is a split simply connected simple group of rank > 1 was obtained by H. Matsumoto (1966, 1969), by using the known cases $\mathbf{SL}_3$ and $\mathbf{Sp}_4$. The congruence subgroup problem, as well as its connection to Moore's theory of universal coverings of $G(\mathbf{A}_k^f)$, is discussed in Séminaire Bourbaki [S123, Œ 77(1967)].

10.1.4. The paper [S143, Œ 86(1970)] is about the S-congruence subgroup problem for $\mathbf{SL}_2$. If $\#S \geqslant 2$, the answer is almost positive: the congruence kernel C^S is a finite cyclic group whose order is at most equal to the number of roots of unity in k; if k is not totally imaginary one has $C^S = 1$: the problem has a positive answer. If $\#S = 1$, the problem has a quite negative answer: C^S is an infinite group. The proof is very interesting. In the case $\#S = 1$, it uses number theory, while in the case $\#S > 1$ it uses topology. We shall now provide some of the details of this proof.

In the case $\#S \geqslant 2$, Serre shows that C^S is contained in the centre of $\widehat{G(k)}$ and then makes use of a theory of C. Moore in order to determine it, and in particular to show that it is finite and cyclic. The finiteness of C^S has some important consequences. For instance, given an S-arithmetic subgroup $N \subset \mathbf{SL}_2(O)$, a field

K of characteristic zero and a linear representation $\rho : N \to G(K)$, there exists a subgroup $N_1 \subset N$, of finite index, such that the restriction of ρ to N_1 is algebraic. This implies that ρ is semisimple. Moreover, for every $k[N]$-module V of finite rank over K, we have $H^1(N, V) = 0$. In particular, when taking for V the adjoint representation, one sees that N is rigid.

If $\#S = 1$, Serre shows that, for most S-arithmetic subgroups N, the group N^{ab} is infinite (this is enough to show that the S-congruence problem has a negative answer). There are three cases: $\mathrm{char}(k) = p > 0$; $k = \mathbf{Q}$; and k an imaginary quadratic field. In each case, there is a contractible "symmetric space" X on which N acts properly, and a study of X/N shows that N^{ab} is infinite (with a few exceptions).

In the case of characteristic $p > 0$, X is the Bruhat–Tits tree. In the case $k = \mathbf{Q}$, X is Poincaré's half-plane. In the case where k is an imaginary quadratic field, X is the hyperbolic 3-space, and the quotient manifold X/N can be compactified by adding to it a finite set of 2-tori (which correspond to elliptic curves with complex multiplication by k); this is a special case of the general compactifications introduced a few years later by Borel and Serre, see Sect. 10.2.2.

10.2. Cohomology of Arithmetic Groups. A locally algebraic group A over a perfect field k is called by Borel–Serre a k-group of type (ALA) if it is an extension of an arithmetic $\mathrm{Gal}(\bar{k}/k)$-group by a linear algebraic group over k. In a joint paper with Borel [S106 (1964)], included in [A. Borel. *Œuvres*, *Collected Papers*], it is proved that for k a number field and S a finite set of places of k, the mapping $H^1(k, A) \to \prod_{v \notin S} H^1(k_v, A)$ is proper, i.e., the inverse image of any point is finite. This result, applied to $A = \mathrm{Aut}(G)$, implies the finiteness of the number of classes of k-torsors of a linear group G which are isomorphic locally everywhere to a given k-torsor.

10.2.1. Let k be a global field and S a finite set of places of k. Let L be a linear, reductive, algebraic group defined over k. In [S139, Œ 83(1969)], [S148 (1971)], and [S149, Œ 88(1971)], Serre undertakes the study of the cohomology of the S-arithmetic subgroups Γ which are contained in $L(k)$. For this purpose, the group Γ is viewed as a discrete subgroup of a finite product $G = \prod G_\alpha$ of (real or ultrametric) Lie groups. The main tool is provided by the Bruhat–Tits buildings associated to the v-adic Lie groups $L(k_v)$, for $v \in S \setminus S_\infty$. The most important contributions include bounds for the cohomological dimension $\mathrm{cd}(\Gamma)$, finiteness properties, and several results relating to the Euler–Poincaré characteristic $\chi(\Gamma)$ and its relations with the values of zeta functions at negative integers (generalizing the well-known formula $\chi(\mathbf{SL}_2(\mathbf{Z})) = \zeta(-1) = -1/12$).

10.2.2. Borel and Serre [S144, Œ 90(1970)] prove that if G is a connected, reductive, linear algebraic group defined over $\mathbf{Q}$, which does not have non-trivial characters, it is possible to associate to G a contractible manifold with corners $\overline{X}$, whose interior, X, is a homogeneous space of $G(\mathbf{R})$ isomorphic to a quotient $G(\mathbf{R})/K$ for a maximal compact subgroup K of $G(\mathbf{R})$. Its boundary $\partial\overline{X}$ has the same homotopy type as the Tits building X of G (i.e. the simplicial complex whose faces correspond to the k-parabolic subgroups of G). An arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$ acts properly on $\overline{X}$ and the quotient $\overline{X}/\Gamma$ is compact; this gives a compactification of X/Γ which is often called the "Borel–Serre compactification". If Γ is torsion-free, then the cohomology of Γ is isomorphic to that of $\overline{X}/\Gamma$ and certain duality relations are fulfilled. In particular, the cohomological dimension of Γ is given by $\mathrm{cd}(\Gamma) = \dim(X) - \mathrm{rg}_{\mathbf{Q}}(G)$. If $G = \mathbf{SL}_n$, the space $\overline{X}$ is, essentially, a space already defined by C.L. Siegel; it is obtained by attaching boundary points and ideal points to X by means of the reduction theory of quadratic forms.

10.2.3. Borel and Serre investigated in [S152, Œ 91(1971)] the cohomology of S-arithmetic groups. Let G be a semisimple algebraic group over an algebraic number field k and let $\Gamma \subset G(k)$ be an S-arithmetic subgroup. The group Γ is a discrete subgroup of $\prod_{v\in S} G(k_v)$. Let X_S be the space defined by:

$$X_S = \overline{X}_\infty \times \prod_{v\in S\setminus S_\infty} X_v.$$

Here X_v is the Bruhat–Tits building of G over k_v and $\overline{X}_\infty$ is the variety with corners associated to the algebraic group $\mathrm{Res}_{k/\mathbf{Q}}(G)$, obtained by restriction of scalars, see above. The group $G(k)$ and, *a fortiori*, the group Γ, acts on X_S. Moreover Γ acts properly and the quotient X_S/Γ is compact. The study of the cohomology of Γ is thus reduced to that of X_S/Γ. In order to go further, Borel and Serre need some information on the cohomology with compact support of each X_v; they obtain it by compactifying X_v, the boundary being the Tits building of $G(k_v)$, endowed with a suitable topology. Their main results may be summarized as follows:

> Let $d = \dim(X_\infty)$, $m = \dim(X_S) - \ell = d - \ell + \sum_{v\in S\setminus S_\infty} \ell_v$, where ℓ, ℓ_v denote the rank of G over k and k_v, respectively. Assume that Γ is torsion-free. Then
>
> $$H^q(\Gamma, M) \simeq H_{m-q}(\Gamma, I_S \otimes M),$$
>
> for every Γ-module M and for every integer q, the dualizing module $I_S = H_c^m(X_S, \mathbf{Z})$ being free over $\mathbf{Z}$. Moreover, $\mathrm{cd}(\Gamma) = m$ and the group $H^q(\Gamma, \mathbf{Z}(\Gamma))$ is equal to 0 if $q \neq m$ and is equal to I_S if $q = m$.

The proofs can be found in the two papers by Borel–Serre [S159 (1973)], [S172 (1976)], which are reproduced in [A. Borel. *Œuvres*, *Collected Papers*, no. 98 and no. 105]; see also the survey [S190, Œ 120(1979)].

11 Arithmetic of Algebraic Varieties

11.1. Modular Curves. The three publications [S142 (1970)], [S170 (1975)], and [S183 (1977)] correspond to lectures delivered by Serre in the Séminaire Bourbaki. They were very popular in the seventies as introductory texts for the study of the arithmetic of modular curves.

11.1.1. The first lecture [S142 (1970)] deals with a theorem of Y. Manin (1969) according to which, given a number field K, an elliptic curve E defined over K, and a prime number p, the order of the p-component of the torsion group $E_{\mathrm{tor}}(K)$ is bounded by an integer depending only on K and p. The proof relies on a previous result by V.A. Demjanenko and Manin on the finiteness of the number of rational points of certain algebraic curves; this is applied afterwards to the modular curve $X_0(N)$, where $N = N(p, K)$.

11.1.2. The second lecture [S170 (1975)] was written jointly with B. Mazur. Its purpose is to present results of A. Ogg on the cuspidal group of the Jacobian of the modular curve $X_0(N)$ and some of the results of Mazur on the rational points of this curve. In it, we find the modular interpretation of the modular curve, the definition of the Hecke operators as correspondences acting on it, the study of the Eisenstein ideal, a study of the Néron model of the Jacobian of the modular curve, a study of the regular model of the modular curve, and so on.

11.1.3. The third lecture [S183 (1977)] explains the results of Mazur on the Eisenstein ideal and the rational points of modular curves (1978), and on the rational isogenies of prime degree (1978). The main theorem is that, if N is a prime not belonging to the set $\{2, 3, 5, 7, 11, 13, 17, 19, 37, 43, 67, 163\}$, then the modular curve $X_0(N)$ has no rational points other than the cusps. As an application, one obtains the possible structures for the rational torsion groups $E_{tor}(\mathbf{Q})$ of the elliptic curves defined over $\mathbf{Q}$. In many aspects, the above work paved the way for G. Faltings' proof of the Mordell–Weil theorem (1983).

11.1.4. The paper [S237 Œ 159(1993)], written with T. Ekedahl, gives a long list of curves of high genus whose Jacobian is completely decomposable, i.e., isogenous to a product of elliptic curves. They ask whether it is true that, for every genus $g > 0$, there exists a curve of genus g whose Jacobian is completely decomposed or, on the contrary, whether the genus of a curve whose Jacobian is completely decomposable is bounded. (The second question has a negative answer in characteristic $p > 0$.) The examples are constructed either by means of modular curves or as coverings of curves of genus 2 or 3. The highest genus obtained is 1297.

11.1.5. Let p be a prime number. In the paper [S254 Œ 170(1997)], Serre determines the asymptotic distribution of the eigenvalues of the Hecke operators T_p on spaces of modular forms when the weight or the level varies. More precisely, let T_p denote the Hecke operator associated to p acting on the space $S(N, k)$ of cusp forms of weight k for the congruence group $\Gamma_0(N)$, with $\gcd(N, p) = 1$, and let $T'_p = T_p/p^{(k-1)/2}$. By Deligne's theorem on the Ramanujan–Petersson conjecture, the eigenvalues of the operator T'_p belong to the interval $\Omega = [-2, 2]$. Let us consider sequences of pairs of integers (N_λ, k_λ) such that k_λ is even, $k_\lambda + N_\lambda \to \infty$ as $\lambda \to \infty$, and p does not divide N_λ. The main theorem proved in the paper states that the family $x_\lambda = x(N_\lambda, k_\lambda)$ of eigenvalues of the $T'_p(N, k)$ is equidistributed in the interval Ω with respect to a measure μ_p which is given by an explicit formula, similar (but not identical) to the Sato–Tate measure. In fact, in the paper, we find measures μ_q which are defined for every $q \geqslant 1$ and have the property that $\lim_{q\to\infty} \mu_q = \mu_\infty$, where μ_∞ is the Sato–Tate measure. In order to give an interpretation of μ_q, Serre identifies Ω with a subset of the spectrum of the automorphism group $G = \mathrm{Aut}(A)$ of a regular tree of valency $q+1$, which is a locally compact group with respect to the topology of simple convergence. Then, μ_q is the restriction to Ω of the Plancherel measure of G. Several interesting consequences are derived from the equidistribution theorem. For instance, it is shown that the maximum of the dimension of the $\mathbf{Q}$-simple factors of the Jacobian $J_0(N)$ of the modular curve $X_0(N)$ tends to infinity as $N \to \infty$. In particular, there are only finitely many integers $N \geqslant 1$ such that $J_0(N)$ is isogenous over $\mathbf{Q}$ to a product of elliptic curves, as was already stated in [S237, Œ 159(1993)].

11.2. Varieties Over Finite Fields. Let $q = p^e$, with p a prime number and $e \geqslant 1$, and let $\mathbf{F}_q$ be a finite field with q elements. The numbers N_{q^r}, $r \geqslant 1$, of rational points over $\mathbf{F}_{q^r}$ of non-singular projective varieties defined over $\mathbf{F}_q$ are encapsulated in their zeta function. One of the major achievements of A. Grothendieck and his school was to provide the tools for the proof of the Weil conjectures (1949), relative to the nature of these functions: cohomological interpretation, rationality, functional equation and the so-called Riemann hypothesis. Serre had a profound influence on the process. An account of the landmark paper by P. Deligne (1974) on the proof of the Riemann hypothesis for the zeta function of a non-singular variety defined over a finite field can be found in [S166 (1974)].

The paper [Œ 117(1978)] contains a report by Serre on Deligne's work, upon request by the Fields Medal Committee. Deligne was awarded the Fields Medal in 1978.

A. Grothendieck was awarded the Fields Medal in 1966, together with M. Atiyah, P. Cohen and S. Smale. Serre's original report, written in 1965, on the work of Grothendieck and addressed to the Fields Medal Committee was reproduced much later, in [S220 (1989)].

Among many other applications, Weil conjectures imply good estimations for certain exponential sums, since these sums can be viewed as traces of Frobenius endomorphisms acting on the cohomology of varieties over finite fields. The results of Deligne on this subject were explained by Serre in [S180, Œ 111(1977)].

11.3. Number of Points of Curves Over Finite Fields. Let C be an absolutely irreducible non-singular projective curve of genus g defined over $\mathbf{F}_q$. After the proof of the Riemann hypothesis for curves, due to Weil (1940–1948), it was known that the number $N = N(C)$ of the rational points of C over $\mathbf{F}_q$ satisfies the inequality $|N - (q+1)| \leqslant 2gq^{1/2}$. Several results due to H. Stark (1973), Y. Ihara (1981), and V.G. Drinfeld and S.G. Vladut (1983) showed that Weil's bound can often be improved. On the other hand, it was of interest for coding theory to have curves of low genus with many points.

In the papers [S201, Œ 128(1983)] and [S200, Œ 129(1983)], Serre expands the results of the previous authors and introduces a systematic method to obtain more precise bounds, based on Weil's "explicit formula".

Let $N_q(g)$ be the maximum value of $N(C)$ as C runs through all curves of genus g defined over $\mathbf{F}_q$. The value $N_q(1)$ was already known; for most q's it is equal to $q+1+\lfloor 2q^{1/2}\rfloor$; for the others it is $q+\lfloor 2q^{1/2}\rfloor$. Serre obtains the exact value of $N_q(2)$ for every q. It is not very different from Weil's bound.

If $A(q)=\limsup_{g\to\infty} N_g(q)/q$, then Drinfeld–Vladut proved that

$$A(q)\leqslant q^{1/2}-1$$

for every q, and Ihara showed that $A(q)=q^{1/2}-1$ if q is a square. Serre proves that $A(q)>0$ for all q, more precisely $A(q)\geqslant c\cdot\log(q)$ for some absolute $c>0$. His proof uses class field towers, like in Golod–Shafarevich.

These papers generated considerable interest in determining the actual maximum and minimum of the number of points for a given pair (g,q).

11.3.1. Kristin Lauter obtained improvements on the bounds for the number of rational points of curves over finite fields, along the lines of [S201, Œ 128(1983)] and [S200, Œ 129(1983)]. In her papers [S265 (2001)], [S267 (2002)], we find appendices written by Serre. The appendix in [S267 (2002)] is particularly appealing. It gives an equivalence between the category of abelian varieties over $\mathbf{F}_q$ which are isogenous over $\mathbf{F}_q$ to a product of copies of an ordinary elliptic curve E defined over $\mathbf{F}_q$ and the category of torsion-free R_d-modules of finite type, where R_d denotes the ring of integers in the quadratic field of discriminant d, being $\#E(\mathbf{F}_q)=q+1-a$, $d=a^2-4q$, and under the assumption that d is the discriminant of an imaginary quadratic field. Polarizations on these abelian varieties correspond to positive definite hermitian forms on R-modules. Thus, in the cases where there is no indecomposable positive definite hermitian module of discriminant 1, one obtains the non-existence of curves whose Jacobian is of that type. And, conversely, if such a hermitian module exists, one obtains a principally polarized abelian variety; if furthermore its dimension is 2, this abelian variety is a Jacobian and one gets a curve whose number of points is $q+1-2a$; a similar (but less precise) result holds for genus 3: one finds a curve with either $q+1-3a$ or $q+1+3a$ points. Particular results on the classification of these modules in dimensions 2 and 3, due to D.W. Hoffmann (1991), and a procedure for gluing isogenies are used to determine the existence or non-existence of certain polarized abelian varieties, useful in their turn to show that for all finite fields $\mathbf{F}_q$ there exists a genus 3 curve over $\mathbf{F}_q$ such that its number of rational points is within 3 of the Serre–Weil upper or lower bound.

11.3.2. In a letter published in [S230, Œ 155(1991)], Serre answered a problem posed by M. Tsfasman at Luminy on the maximal number of points of a hypersurface defined over a finite field. On the hypersurface, no hypothesis of irreducibility or of non-singularity is made. Serre shows that the number N of zeros of a homogeneous polynomial $f = f(X_0, \dots, X_n)$ in $\mathbf{P}_n(\mathbf{F}_q)$ of degree $d \leqslant q + 1$ is at most $dq^{n-1} + p_{n-2}$, where $p_n = q^n + q^{n-1} + \cdots + 1$ is the number of points in the projective space $\mathbf{P}_n(\mathbf{F}_q)$. The result has been widely used in coding theory.

11.4. Diophantine Problems. Certain classical methods of transcendence based on the study of the solutions of differential equations, mainly due to Th. Schneider, were transposed to the p-adic setting by S. Lang (1962, 1965). In the Séminaire Delange–Pisot–Poitou [S124 (1967)], Serre studies the dependence of p-adic exponentials avoiding the use of differential equations.

11.4.1. The book *Lectures on the Mordell–Weil Theorem* [S203, MW(1984)] arose from the notes taken by M. Waldschmidt of a course taught by Serre (1980–1981), which were translated and revised with the help of M. Brown.

The content of the lectures was: heights, Néron–Tate heights on abelian varieties, the Mordell–Weil theorem on the finiteness generation of the rational points of any abelian variety defined over a number field, Belyi theorem characterizing the non-singular projective complex curves definable over $\overline{\mathbf{Q}}$, Chabauty and Manin–Demjanenko theorems on the Mordell conjecture (previous to Faltings' theorem (1983)), Siegel's theorem on the integral points on affine curves, Baker's effective forms of Siegel's theorem, Hilbert's irreducibility theorem and its applications to the inverse Galois problem, construction of elliptic curves of large rank, sieve methods, Davenport–Halberstam's theorem, asymptotic formulas for the number of integral points on affine varieties defined over number fields, and the solution to the class number 1 problem by using integral points on modular curves.

11.4.2. The paper [S189, Œ 122(1979)] is an appendix to a text by M. Waldschmidt on transcendental numbers (1970). It contains several useful properties of connected commutative algebraic groups, defined over a field k of characteristic zero. They concern the following: the existence of smooth projective compactifications; quadratic growth, at most, of the height function (when k is an algebraic extension of $\mathbf{Q}$), and uniformization by entire functions of order $\leqslant 2$ when $k = \mathbf{C}$.

11.4.3. The publication [S194 (1980)] reproduces two letters addressed to D. Masser. The questions concern some of Masser's results on the linear independence of periods and pseudo-periods of elliptic functions (1977). In the first letter, Serre studies independence properties of the fields of ℓ-division points of elliptic curves defined over an algebraic number field and with complex multiplication by different quadratic imaginary number fields. In the second letter, Serre proves that, under some reasonable assumptions, the degree of the field generated by the ℓ-division points of the product of such elliptic curves is as large as possible for almost all the primes ℓ.

12 Field Theory

The paper [S195, Œ 123(1980)] reproduces a letter of Serre answering a question raised by J.D. Gray about Klein's lectures on the icosahedron. The icosahedral group $G = A_5$ acts on a curve X of genus zero; by extending the field of definition k of X and G to an algebraic closure, one obtains an embedding of G in the projective linear group $\mathbf{PGL}_2$. Moreover, the field k must contain $\sqrt{5}$. The quotient X/G is isomorphic to $\mathbf{P}_1$. If z is a k-point of X/G, its lifting to X generates a Galois extension k' of k whose Galois group is a subgroup of G. Serre explains that the main question posed by Hermite and Klein turns out to be whether one obtains all Galois extensions of k with Galois group G in this way. He then shows that the answer to this question is "almost" yes. Suppose that k'/k is a Galois extension with Galois group G. Serre uses a descent method and works with twisted curves $X_{k'}$. The curves $X_{k'}$ are controlled by a quaternion algebra $H_{k'}$. To go from $X_{k'}$ to $H_{k'}$, Serre follows two procedures: either using the non-trivial element of $H^2(G, \mathbf{Z}/2\mathbf{Z})$, which corresponds to the binary icosahedral group, or considering the trace form $\mathrm{Tr}(z^2)$ in a quintic extension k_1/k defining k'/k. Then he shows that k'/k comes from a covering $X \to X/G$ if and only if $X_{k'}$ has a rational point over k, if and only if the class of H'_k in the Brauer group of k equals the sum of $(-1, -1)$ and the Witt invariant of the quadratic form $\mathrm{Tr}(z^2)$, on the subspace of k_1 of elements of trace zero. Moreover, these conditions are equivalent to the fact that k' can be generated by the roots of a quintic equation of the form $X^5 + aX^2 + bX + c = 0$, which is consistent with old results of Hermite and Klein.

12.0.1. The obstruction associated to a Galois embedding problem, defined by a Galois extension L/K and by a central extension of the group $\mathrm{Gal}(L/K)$, is given by a cohomology class, the vanishing of which characterizes the solvability of the problem. When the kernel of the central extension is the cyclic group C_2 of order 2, the cohomology class can be identified with an element of $\mathrm{Br}_2(K) \simeq H^2(K, C_2)$ (assuming that the characteristic is not 2). In a paper dedicated to J.C. Moore [S204, Œ 131(1984)], Serre gives a formula relating the obstruction to certain Galois embedding problems to the second Stiefel–Whitney class of the trace form $\mathrm{Tr}(x^2)$. Through the use of Serre's formula, N. Vila (1984, 1985) proved that the non-trivial double covering $\widetilde{A}_n = 2 \cdot A_n$ of the alternating group A_n is the Galois group of a regular extension of $\mathbf{Q}(T)$, for infinitely many values of $n \geqslant 4$. The result was extended to all $n \geqslant 4$ by J.-F. Mestre (1990) cf. [S235, TGT(1992)]. Explicit solutions to solvable embedding problems of this type were later obtained by T. Crespo.

12.0.2. In his report on Galois groups over $\mathbf{Q}$ presented to the Séminaire Bourbaki [S217, Œ 147(1988)], Serre provides a summary of the status of the inverse Galois problem; that is, of the question of whether all finite groups are Galois groups of an equation with rational coefficients. He mentions the solution of the problem in the solvable case due to I. Shafarevich (1954) and its improvements by J. Neukirch (1979). He gives Hilbert's realizations of the symmetric and alternating groups as Galois groups over $\mathbf{Q}$ by means of Hilbert's irreducibility theorem. And, in the most detailed part of the exposition, he explains the rigidity methods of H. Matzat (1980) and J.G. Thompson (1984), and presents a list of the simple groups known at that moment to be Galois over **Q**. As another type of example, he considers the realization of certain central extensions of simple groups, which had recently been obtained thanks to his $\mathrm{Tr}(x^2)$ formula [S204, Œ 131(1984)].

12.0.3. The papers [S226, Œ 151(1990)] and [S227, Œ 152(1990)] are related to the results of Mestre mentioned above. The first one is about lifting elements of odd order from A_n to $\widetilde{A}_n$; if one has several elements and their product is equal to 1, what is the product of their liftings: 1 or -1? In the second paper (which may be viewed as a geometrization of the first one), Serre considers a ramified covering $\pi : Y \to X$ of curves, in which all the ramification indices are odd. He gives a formula relating several cohomological invariants of this covering; here the behaviour of the theta characteristics of X under π^* plays an essential role. He also asks whether there is a general formula including those in [S204, Œ 131(1984)] and [S227, Œ 152(1990)]. This was done later by H. Esnault, B. Kahn and E. Viehweg (1993).

12.0.4. The course given by Serre [S236, Œ 156(1992)] focuses on the Galois cohomology of pure transcendental extensions. Suppose that K is a field endowed with a discrete valuation, v, of residue field k. Let C denote a discrete $\mathrm{Gal}(K_s/K)$-module, unramified at v and such that $nC = 0$ for some integer $n > 0$, coprime to the characteristic of K. Given a cohomology class $\alpha \in H^i(K, C)$, one defines the notions of a residue of α at v, a pole of α at v, and a value $\alpha(v)$. When $K = k(X)$ is the function field of a smooth, connected projective curve defined over k, there is a residue formula and an analogue of Abel's theorem. The theory is applied to the solution of specialization problems of the Brauer group of K in the Brauer group of k. If $x \in X(K)$, and $\alpha \in \mathrm{Br}_n(K)$, then $\alpha(x) \in \mathrm{Br}_n(k)$, whenever x is not a pole of α. Serre looks at the function $\alpha(x)$ and, in particular, at its vanishing set $V(\alpha)$. In [S225, Œ 150(1990)], he deals with the case $K = \mathbf{Q}(T_1, \ldots, T_r)$, for $n = 2$. The results are completed with asymptotic estimations on the number of zeros of α obtained by sieving arguments; they depend on the number of **Q**-irreducible components of the polar divisor of α. One of the questions raised in this paper ("how often does a conic have a rational point?") was solved later by C. Hooley (1993) and C.R. Guo (1995): the upper bound given by the sieve method has the right order of magnitude.

12.0.5. *Cohomological Invariants in Galois Cohomology* [S269, CI(2003)] is a book co-authored by S. Garibaldi, A. Merkurjev and J.-P. Serre. The algebraic invariants discussed in it are the Galois cohomology analogues of the characteristic classes of topology, but here the topological spaces are replaced by schemes $\mathrm{Spec}(k)$, for k a field.

The text is divided in two parts. The first one consists of an expanded version of a series of lectures given by Serre at UCLA in 2001, with notes by S. Garibaldi. The second part is due to Merkurjev with a section by Garibaldi; we shall not discuss it here. In Chap. I, Serre defines a quite general notion of invariant to be applied throughout the book to several apparently disparate situations. Given a ground field k_0 and two functors

$$A : \mathrm{Fields}_{/k_0} \to \mathrm{Sets} \quad \text{and} \quad H : \mathrm{Fields}_{/k_0} \to \text{Abelian Groups},$$

an H-invariant of A is defined as a morphism of functors

$$a : A \to H.$$

Here $\mathrm{Fields}_{/k_0}$ denotes the category of field extensions k of k_0. Examples of functors A are:

- $k \mapsto \mathrm{Et}_n(k)$, the isomorphism classes of étale k-algebras of rank n;
- $k \mapsto \mathrm{Quad}_n(k)$, the isomorphism classes of non-degenerate quadratic forms over k of rank n;
- $k \mapsto \mathrm{Pfister}_n(k)$, the isomorphism classes of n-Pfister forms over k of rank equal to n.

Examples of functors H are provided by the abelian Galois cohomology groups $H^i(k, C)$ and their direct sum, $H(k, C)$, where C denotes a discrete $\mathrm{Gal}(k_0^s/k_0)$-module; or by the functor $H(k) = W(k)$, where $W(k)$ stands for the Witt ring of non-degenerate quadratic forms on k.

The aim of the lectures is to determine the group of invariants $\mathrm{Inv}(A, H)$. In the background material of the book concerning Galois cohomology, we find the notion of the residue of a cohomology class of $H^i(K, C)$ at a discrete valuation v of a field K, as well as the value at v for those cohomology classes with residue equal to zero. Basic properties of restriction and corestriction of invariants are obtained, in perfect analogy with the case of group cohomology. An important tool is the notion of versal torsor, which plays an analogous role to that of the universal bundle in topology: an invariant is completely determined by its value on a versal G-torsor. These techniques allow the determination of the mod 2 invariants for quadratic forms, hermitian forms, rank n étale algebras, octonions or Albert algebras, when $\mathrm{char}(k_0) \neq 2$. In particular, the mod 2 invariants of rank n étale algebras make up a free $H(k_0)$-module whose basis consists of the Stiefel–Whitney classes w_i, for $0 \leqslant i \leqslant [n/2]$. This gives a new proof of Serre's earlier formula on this subject [S204, Œ 131(1984)], as well as its generalization by B. Kahn (1984). Similarly, the $W(k_0)$-module $\mathrm{Inv}(S_n, W)$ is free of finite rank, with basis given by the Witt classes of the first $[n/2]$ exterior powers of the trace form. Among other results, one finds an explicit description of all possible trace forms of rank $\leqslant 7$ and an application of trace forms to the study of Noether's problem, which we recall in what follows.

Given a finite group G, the property $\mathrm{Noe}(G/k_0)$ means that there exists an embedding $\rho : G \to \mathbf{GL}_n(k_0)$ such that, if K_ρ is the subfield of $k_0(X_1, \ldots, X_n)$ fixed by G, then K_ρ is a pure transcendental extension of k_0. Deciding whether $\mathrm{Noe}(G/k_0)$ is true is the Noether problem for G and k_0. Serre proves that Noether's problem has a negative answer for $\mathbf{SL}(2, \mathbf{F}_7)$, $2 \cdot A_6$ or the quaternion group Q_{16} of order 16. The book includes also several letters; one of these is a letter from Serre to R.S. Garibaldi, dated in 2002, in which he explains his motivations.

12.0.6. Let k be a field of characteristic different from 2. The norm form of any quaternion algebra defined over k is a 2-fold Pfister form. In [S278 (2006)], M. Rost, J.-P. Serre and J.-P. Tignol study the trace form $q_A(x) = \mathrm{Trd}_A(x^2)$ of a central simple algebra A of degree 4 over k, under the assumption that k contains a primitive 4^{th} root of unity. They prove that $q_A = q_2 + q_4$, in the Witt ring of quadratic forms over k, were q_2 and q_4 are uniquely determined 2-fold and 4-fold Pfister forms, respectively. The form q_2 corresponds to the norm form of the quaternion algebra which is equivalent to $A \otimes_k A$ in the Brauer group of k. Moreover, A is cyclic if and only if q_4 is hyperbolic. The images of the forms q_j in $H^j(k, \mathbf{Z}/2\mathbf{Z})$ yield cohomological invariants of $\mathbf{PGL}_4$, since the set $H^1(k, \mathbf{PGL}_4)$ classifies the central simple k-algebras of degree 4.

13 Galois Representations

Serre has studied Galois representations (especially ℓ-adic representations) in several books and papers. His pioneering contributions to these topics have broken new ground and have profoundly influenced their research in the last decades.

13.1. Hodge–Tate Modules. The paper [S129, Œ 72(1967)] is based on a lecture delivered by Serre at a Conference on Local Fields held in Driebergen (The Netherlands). Take as the ground field a local field of characteristic zero whose residue field is of characteristic $p > 0$, and let $\mathbf{C}_p$ be the completion of an algebraic closure $\overline{K}$ of K. If T is the Tate module associated to a p-divisible group, defined over the ring of integers of K, a deep result of Tate states that $\mathbf{C}_p \otimes T$ has a decomposition analogous to the Hodge decomposition for complex cohomology. This gives strong restrictions on the image G of $\mathrm{Gal}(\overline{K}/K)$ in $\mathrm{Aut}(T)$. For instance, if the action of G is semisimple, then the Zariski closure of G contains a p-adic Mumford–Tate group. Under some additional hypotheses, Serre shows that the group G is open in $\mathrm{Aut}(T)$. This applies in particular to formal groups of dimension 1, without formal complex multiplication.

13.1.1. The topic of Hodge–Tate decompositions was also considered in the book [S133, McGill(1968)], which we will discuss in a moment.

13.1.2. In [S191, Œ 119(1979)], it is shown that the inertia subgroup of a Galois group acting on a Hodge–Tate module V over a local field is almost algebraic, in the sense that it is open in a certain algebraic subgroup H_V of the general linear group $\mathbf{GL}_V$. In two important cases, Serre determines the structure of the connected component H_V^0 of H_V. In the commutative case, H_V^0 is a torus. If the weights of V are reduced to 0 and 1, the simple factors of H_V^0 are of classical type: A_n, B_n, C_n, D_n.

13.2. Elliptic Curves and ℓ-adic Representations. Over the years, Serre has given several courses on elliptic curves. Three of these were at the Collège de France

[S115, Œ 67(1965)], [S122, Œ 71(1966)], [S153, Œ 93(1971)] and one at McGill University (Montreal), in 1967. Abundant material was presented in these lectures. Most of it was published soon after in the papers [S118, Œ 70(1966)], [S154, Œ 94(1972)] and in the book *Abelian ℓ-adic representations and elliptic curves* [S133, McGill(1968)].

The course [S115, Œ 67(1965)] covered general properties of elliptic curves, theorems on the structure of their endomorphism ring, reduction of elliptic curves, Tate modules, complex multiplication, and so on. Let E be an elliptic curve defined over a field k and let ℓ be a prime different from char(k). The Tate modules $T_\ell(E) = \varprojlim E[\ell^n](k_s)$ are special cases of the ℓ-adic homology groups associated to algebraic varieties. The Galois group $\mathrm{Gal}(k_s/k)$ acts on $T_\ell(E)$ and on the $\mathbf{Q}_\ell$-vector space $V_\ell(E) = \mathbf{Q}_\ell \otimes T_\ell(E)$. We may consider the associated Galois ℓ-adic representation $\rho_\ell : \mathrm{Gal}(k_s/k) \to \mathrm{Aut}(T_\ell) \simeq \mathbf{GL}_2(\mathbf{Z}_\ell)$. The image G_ℓ of ρ_ℓ is an ℓ-adic Lie subgroup of $\mathrm{Aut}(T_\ell)$. We shall denote by $\mathfrak{g}_\ell$ the Lie algebra of G_ℓ. The Galois extension associated to G_ℓ is obtained by adding the coordinates of the points of $E(\bar{k})$ of order a power of ℓ to the field k.

Suppose that k is an algebraic number field and that the elliptic curve E has complex multiplication. Thus, there exist an imaginary quadratic field F and a ring homomorphism $F \to \mathbf{Q} \otimes \mathrm{End}_k(E)$. Then the Galois group G_ℓ is abelian whenever $F \subset k$, and is non-abelian otherwise. If $F \subset k$, the action of $\mathrm{Gal}(\bar{k}/k)$ on $T_\ell(E)$ is given by a Grössencharakter whose conductor has its support in the set of places of bad reduction of E; this result is due to M. Deuring. The usefulness of elliptic curves with complex multiplication consists in the fact that they provide an explicit class field theory for imaginary quadratic fields.

13.2.1. A short account of the classical theory of complex multiplication can be found in [S128, Œ 76(1967)].

13.2.2. By using fiber spaces whose fibers are products of elliptic curves with complex multiplication, Serre [S107, Œ 63(1964)] constructed examples of non-singular projective varieties defined over an algebraic number field K which are Galois conjugate but have non-isomorphic fundamental groups. In particular, although they have the same Betti numbers, they are not homeomorphic.

13.2.3. The paper [S118, Œ 70(1966)] is about the ℓ-adic Lie groups and the ℓ-adic Lie algebras associated to elliptic curves defined over an algebraic number field k and without complex multiplication. The central result is that $\mathfrak{g}_\ell$ is "as large as possible", namely, it is equal to $\mathrm{End}(V_\ell)$, when the ground field is $\mathbf{Q}$. In the proof, Serre uses a wide range of resources: Lie algebras and ℓ-adic Lie groups; Hasse–Witt invariants of elliptic curves; pro-algebraic groups; the existence of canonical liftings of ordinary curves in characteristic p; the Lie subalgebras of the ramification groups; Chebotarev's density theorem, as well as class field theory, Hodge–Tate theory, and so on. Serre also observes that, if a conjecture of Tate on Galois actions on the Tate modules is true, then the determination of the Lie algebra $\mathfrak{g}_\ell$ can be carried out for any algebraic number field. Tate's conjecture was proved almost two decades later by G. Faltings (1983) in a celebrated paper in which he also proved two more conjectures, one due to L.J. Mordell and the other due to I. Shafarevich. For these results, Faltings was awarded the Fields Medal in 1986.

In the same paper [S118, Œ 70(1966)], Serre shows that the set of places of k at which a curve E, without complex multiplication, has a supersingular reduction is of density zero in the set of all the places of k. This does not preclude the set of these places being infinite. On the contrary, Serre thought that this could well be the case. Indeed, N.D. Elkies (1987) proved that, for every elliptic curve E defined over a real number field, there exist infinitely many primes of supersingular reduction, in agreement with Serre's opinion. (Note that the case of a totally imaginary ground field remains open.) S. Lang and H. Trotter (1976) conjectured an asymptotic formula (which is still unproved) for the frequency of the supersingular primes in the reduction of an elliptic curve E without complex multiplication and defined over $\mathbf{Q}$.

13.2.4. The results of [S118, Œ 70(1966)] were completed in the lecture course [S122, Œ 71(1966)] and in [S133, McGill(1968)].

In Chap. I of [McGill], Serre considers ℓ-adic representations of the absolute Galois group $\mathrm{Gal}(k_s/k)$ of a field k. For k an algebraic number field, he defines the concepts of a rational ℓ-adic representation, and of a compatible system of rational ℓ-adic representations (these notions go back to Y. Taniyama (1957)). He relates the equidistribution of conjugacy classes of Frobenius elements to the existence of some analytic properties for the L-functions associated to compatible systems of rational ℓ-adic representations, a typical example being that of the Sato–Tate conjecture.

In Chap. II, Serre associates to every algebraic number field k a projective family $(S_{\mathfrak{m}})$ of commutative algebraic groups defined over $\mathbf{Q}$. (From the point of view of motives, these groups are just the commutative motivic Galois groups.) For each modulus $\mathfrak{m}$ of k, he constructs an exact sequence of commutative algebraic groups $1 \to T_{\mathfrak{m}} \to S_{\mathfrak{m}} \to C_{\mathfrak{m}} \to 1$, in which $C_{\mathfrak{m}}$ is a finite group and $T_{\mathfrak{m}}$ is a torus. The characters of $S_{\mathfrak{m}}$ are, essentially, the Grössencharakteren of type A_0, in the sense of Weil, of conductor dividing $\mathfrak{m}$. They appear in the theory of complex multiplication.

In Chap. III, the concept of a locally algebraic abelian ℓ-adic representation is defined. The main result is that such Galois representations come from linear representations, in the algebraic sense, of the family $(S_{\mathfrak{m}})$. When the number field k is obtained by the composition of quadratic fields, it is shown that every semisimple abelian rational ℓ-adic representation is locally algebraic. The proof is based upon transcendence results of C.L. Siegel and S. Lang. Serre observes that the result should also be true for any algebraic number field; this was proved later by M. Waldschmidt (1986), as a consequence of a stronger transcendence result.

In Chap. IV, the results of the previous chapters are applied to the ℓ-adic representations associated to elliptic curves. The main theorem is that, if E is an elliptic curve over an algebraic number field k, without complex multiplication, then $\mathfrak{g}_\ell = \mathrm{End}(V_\ell)$. The proof turns out to be a clever combination of a finiteness theorem due to Shafarevich together with the above mentioned results on abelian and locally algebraic ℓ-adic representations of k.

Serre also proves that, if E is an elliptic curve over an algebraic number field k such that its j-invariant is not an algebraic integer of k, then the group $G := \mathrm{Im}\rho$, where $\rho = \prod \rho_\ell$, is open in $\prod \mathbf{GL}_2(\mathbf{Z}_\ell)$. Later, Serre would eliminate the condition regarding the modular invariant j (see below).

It is also proved in [McGill] that, if E, E' are elliptic curves defined over an algebraic number field k, whose invariants $j(E)$, $j(E')$ are not algebraic integers and whose $\mathrm{Gal}(\overline{k}/k)$-modules $V_\ell(E)$, $V_\ell(E')$ are isomorphic, then E and E' are isogenous over k. The result is a special case of the Tate conjecture proved later by G. Faltings (1983).

13.2.5. The above results were improved in the seminal paper *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques* [S154, Œ 94(1972)], which is dedicated to André Weil. The main theorem states that, if E is an elliptic curve defined over an algebraic number field k, which does not have complex multiplication, then $G_\ell = \mathrm{Aut}(T_\ell(E))$, for almost all ℓ. In particular, we have $\mathrm{Gal}(k[E_\ell]/k) \simeq \mathbf{GL}_2(\mathbf{F}_\ell)$, for almost all ℓ. The proof is based upon local results relative to the action of the tame inertia group on the points of finite order of the elliptic curves. This action can be expressed in terms of products of fundamental characters, and the main point is that these exponents have a uniform bound (namely the ramification index of the local field). This boundedness plays a role similar to that of the local algebraicity which had been used in [McGill]. Serre conjectures that similar bounds are valid for higher dimensional cohomology; this has been proved recently, as a by-product of Fontaine's theory.

He also raised several questions concerning the effectiveness of the results. The paper includes many numerical examples in which all the prime numbers for which $\mathrm{Gal}(k[E_\ell]/k) \not\simeq \mathbf{GL}_2(\mathbf{F}_\ell)$ are computed.

In the summary of the course, Serre also mentions that if A is an abelian surface such that $\mathrm{End}(A)$ is an order of a quaternion field D defined over $\mathbf{Q}$ (a so-called "fake elliptic curve") then the group $\rho(\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}))$, where $\rho = \prod \rho_\ell$, is open in $D^*(\mathbf{A}^f)$. This was proved later by M. Ohta (1974), following Serre's guidelines.

13.3. Modular Forms and ℓ-adic Representations. Many arithmetical functions can be recovered from the Fourier coefficients of modular functions or modular forms. In an early contribution at the Séminaire Delange–Pisot–Poitou [S138, Œ 80(1969)], one finds the remarkable conjecture that certain congruences satisfied by the Ramanujan τ function can be explained by the existence, for each prime ℓ, of a 2-dimensional ℓ-adic representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}(V_\ell),$$

unramified away from ℓ, and such that $\mathrm{Tr}(\rho_\ell(F_p)) = \tau(p)$, $\det(\rho_\ell(F_p)) = p^{11}$, for each Frobenius element F_p, at any prime $p \neq \ell$. Assume this conjecture (which was proved a few months later by Deligne (see below)). The ℓ-adic representation ρ_ℓ leaves a lattice of V_ℓ stable, and thus may be viewed as a representation in $\mathbf{GL}_2(\mathbf{Z}_\ell)$. When varying the different primes ℓ, the above representations ρ_ℓ

make up a compatible system of rational ℓ-adic representations of $\mathbf{Q}$, in the sense of [McGill], and the images of ρ_ℓ are almost always the largest possible. The primes for which this does not happen are called the exceptional primes and they are finite in number. More specifically, in the case of the τ function, the exceptional primes are $2, 3, 5, 7, 23, 691$ (this was proved later by Swinnerton-Dyer). For example: $\tau(p) \equiv 1 + p^{11} \pmod{691}$ is the congruence discovered by Ramanujan. As a consequence, the value of $\tau(p)$ mod ℓ cannot be deduced from any congruence on p, if ℓ is a non-exceptional prime.

13.3.1. Serre's conjecture on the existence of ℓ-adic representations associated to modular forms was soon proved by Deligne (1971). This result has been essential for the study of modular forms modulo p, for that of p-adic modular forms, as well as for the work of H.P.F. Swinnerton-Dyer (1973) on congruences. Swinnerton-Dyer's results on this topic were presented by Serre at the Séminaire Bourbaki [S155, Œ 95(1972)].

13.3.2. In the papers [S161, Œ 100(1974)], [S168 (1975)] and [S173, Œ 108(1976)], it is proved that, given a modular form $f = \sum_{n=0}^{\infty} c_n e^{2\pi i n z/M}$ with respect to a congruence subgroup of the full modular group $\mathbf{SL}_2(\mathbf{Z})$, and of integral weight $k \geqslant 1$, for each integer $m \geqslant 1$, the set of integers n which satisfy the congruence $c_n \equiv 0 \pmod{m}$ is of density 1. The proof uses ℓ-adic representations combined with an analytic argument due to E. Landau. Given a cusp form $f = \sum a_n q^n$, $q = e^{2\pi i z}$, without complex multiplication, of weight $k \geqslant 2$, normalized eigenvector of all the Hecke operators and with coefficients in $\mathbf{Z}$, Serre shows that the set of integers n such that $a_n \neq 0$ has a density which is > 0; in particular, the series f is not "lacunary".

13.3.3. Deligne and Serre, in the paper [S162, Œ 101(1974)] dedicated to H. Cartan, prove that every cusp form of weight 1, which is an eigenfunction of the Hecke operators, corresponds by Mellin's transform to the Artin L-function of an irreducible complex linear representation $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}_2(\mathbf{C})$. Moreover, the Artin conductor of ρ coincides with the level of the cusp form (provided it is a newform). In order to prove the theorem, Serre and Deligne construct Galois representations (mod ℓ), for each prime ℓ, of sufficiently small image; this allows them to lift these representations to characteristic zero and to obtain from them the desired complex representation (the proof also uses an average bound on the eigenvalues of the Hecke operators due to Rankin). Note that here the existence of ℓ-adic representations associated to modular forms of weight $k \geqslant 2$ is used to deduce an existence theorem for complex representations associated to weight $k = 1$ modular forms!

The paper became a basic reference on the subject, since it represents a small but non-trivial step in the direction of the Langlands conjectures. In particular, it shows that certain Artin L-functions are entire. Another consequence is that the Ramanujan–Petersson conjecture holds for weight $k = 1$. (For weight $k \geqslant 2$, its truth follows from Deligne's results on Weil's conjectures and on the existence of

ℓ-adic representations associated to cusp forms.) Not long after, the study of modular forms of weight $k = 1$ was illustrated by Serre in [S179, Œ 110(1977)] with many numerical examples, due to Tate.

13.3.4. In 1974, Serre opened the *Journées Arithmétiques* held in Bordeaux with a lecture on Hecke operators (mod ℓ) [S169, Œ 104(1975)]—in those days a fairly new subject for most of the audience. Consider the algebra $\widetilde{M}$ of modular forms (mod ℓ) with respect to the modular group $\mathbf{SL}_2(\mathbf{Z})$. Serre proves that the systems of eigenvalues (a_p) of the Hecke operators T_p, $p \neq \ell$, acting on $\overline{\mathbf{F}}_\ell \otimes \widetilde{M}$ are finite in number. In particular, there exists a weight $k(\ell)$ such that each system of eigenvalues can be realized by a form of weight $\leqslant k(\ell)$; the precise value for $k(\ell)$ was found by Tate. As an illustration, Serre gives a complete list of all the systems (a_p) which occur for the primes $\ell \leqslant 23$. He also raises a series of problems and conjectures which lead, twelve years later, to his own great work [S216, Œ 143(1987)] on modular Galois representations. As is well known, this became a key ingredient in the proof of Fermat's Last Theorem.

13.3.5. In [S178, Œ 113(1977)], Serre and H. Stark prove that each modular form of weight $1/2$ is a linear combination of theta series in one variable, thus answering a question of G. Shimura (1973). The proof relies on the "bounded denominator property" of modular forms on congruence subgroups.

13.3.6. In the long paper entitled *Quelques applications du théorème de densité de Chebotarev* [S197, Œ 125(1981)], one finds a number of precise estimates both for elliptic curves and for modular forms. These estimates are of two types: either they are unconditional, or they depend on the Generalized Riemann Hypothesis (GRH). The work in question is essentially analytic. It uses several different ingredients:

- explicit forms of Chebotarev theorem due to J.C. Lagarias, H.L. Montgomery, A.M. Odlyzko, with applications to infinite Galois extensions with an ℓ-adic Lie group as Galois group;
- properties of ℓ-adic varieties such as the following: the number of points (mod ℓ^n) of an ℓ-adic analytic variety of dimension d is $\mathrm{O}(\ell^{nd})$, for $n \to \infty$;
- general theorems on ℓ-adic representations.

Let us mention two applications:

Given a non-zero modular form $f = \sum a_n q^n$, which is an eigenvalue of all the Hecke operators and is not of type CM (complex multiplication) Serre proves that the series f is not lacunary; more precisely, if $M_f(x)$ denotes the number of integers $n \leqslant x$ such that $a_n \neq 0$, then there exists a constant $\alpha > 0$ such that $M_f(x) \sim \alpha x$ for $x \to \infty$. On the other hand, if $f \neq 0$ has complex multiplication, then there exists a constant $\alpha > 0$ such that $M_f(x) \sim \alpha x/(\log x)^{1/2}$, for $x \to \infty$. In concrete examples, Serre provides estimates for α.

Furthermore, if $E/\mathbf{Q}$ is an elliptic curve without complex multiplication and if we assume (GRH), then there exists an absolute constant c such that the Galois group G_ℓ of the points of the ℓ-division of E is isomorphic to $\mathbf{GL}_2(\mathbf{F}_\ell)$ for every prime $\ell \geqslant c(\log N_E)(\log\log 2N_E)^3$, where N_E denotes the product of all the primes of bad reduction of E.

13.3.7. The Dedekind η function is a cusp form of weight 1/2. In [S208, Œ 139(1985)], a paper which Serre dedicated to R. Rankin, he studies the lacunarity of the powers η^r, when r is a positive integer. If r is odd, it was known that η^r is lacunary if $r = 1, 3$. If r is even, it was known that η^r is lacunary for $r = 2, 4, 6, 8, 10, 14, 26$. Serre proves that, if r is even, the above list is complete. By one of the theorems proved in his Chebotarev paper (see above), this is equivalent to showing that η^r is of CM type only if $r = 2, 4, 6, 8, 10, 14$ or 26. The proof consists of showing that the complex multiplication, if it exists, comes from either $\mathbf{Q}(i)$ or $\mathbf{Q}(\sqrt{-3})$.

13.3.8. The paper [S216, Œ 143(1987)], entitled *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, contains one of Serre's outstanding contributions. In this profound paper, dedicated to Y.I. Manin, Serre formulates some very precise conjectures on Galois representations which extend those made twelve years before in Bordeaux [S169, Œ 104(1975)]. We shall only mention two of these conjectures: conjectures (3.2.3?) and (3.2.4?), known nowadays as Serre's modularity conjectures (or, simply, Serre's modularity conjecture). Let $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}_2(\overline{\mathbf{F}}_p)$ be a continuous irreducible representation of odd determinant.

(3.2.3?) There exists a cusp form f, with coefficients in $\overline{\mathbf{F}}_p$ which is an eigenfunction of the Hecke operators, whose associated representation ρ_f is isomorphic to the original representation ρ.

(3.2.4?) The smallest possible type of the form f of (3.2.3?) is equal to $(N(\rho), k(\rho), \varepsilon(\rho))$, where the level $N(\rho)$ is the Artin conductor of ρ (it reflects the ramification at the primes $\ell \neq p$); the character $\varepsilon(\rho)$ is $\chi^{1-k} \cdot \det(\rho)$, where χ is the ℓ-cyclotomic character; the weight $k(\rho)$ is given by a rather sophisticated formula, which depends only on the ramification at p.

The paper contains numerical examples for $p = 2, 3, 7$ in support of the conjecture; they were implemented with the help of J.-F. Mestre. Some months after the appearance of this paper, and after examining the examples more closely, Serre slightly modified the conjectures for the primes $p = 2, 3$ in the case of Galois representations of dihedral type.

Since their publication, Serre's conjectures have generated abundant literature. They imply Fermat's Last Theorem (and variants thereof) as well as the Shimura–Taniyama–Weil Conjecture (and generalizations of it). As G. Frey (1986) and Serre pointed out, a weak form of conjecture (3.2.4?), known as conjecture *epsilon*, is sufficient to prove that Fermat's Last Theorem follows from the Shimura–Taniyama–Weil conjecture in the semistable case. Conjecture *epsilon* was proved by K. Ribet (1990) in a brilliant study in which he made use of the arithmetical properties of modular curves, Shimura curves and their Jacobians. Once Ribet's theorem was proved, the task of proving Shimura–Taniyama–Weil conjecture in the semistable case would be accomplished five years later by A. Wiles (1995) and R. Taylor and A. Wiles (1995).

Serre's modularity conjecture may be viewed as a first step in the direction of a mod p analogue of the Langlands program. Many people have worked on it. A general proof was presented at a Summer School held at Luminy (France), July 9–20, 2007 (a survey of this work can be found in the expository paper [Ch. Khare (2007)]).

According to the Citation Database MathSciNet, [S216, Œ 143(1987)] is Serre's second most frequently cited paper, the first one being (fittingly) the one he dedicated to A.Weil [S154, Œ 94(1972)].

13.3.9. A summary of Serre's lecture course on Galois representations (mod p) and modular forms (mod p) can be found in [S218, Œ 145(1988)]. In it, Serre relates modular forms modulo p with quaternions. Two letters on this subject, addressed to J. Tate and D. Kazhdan, are collected in [S249, Œ 169(1996)]. In the letter to Tate, Serre formulates a quaternion approach to modular forms modulo a prime p through quaternion algebras. Let D be the quaternion field over $\mathbf{Q}$ ramified only at p and at ∞, and let $D^*(A)$ be the group of the adelic points of the multiplicative group D^*, viewed as an algebraic group over $\mathbf{Q}$. The main result of the letter to Tate is that the systems of eigenvalues (a_ℓ), with $a_\ell \in \overline{\mathbf{F}}_p$, provided by the modular forms (mod p) coincide with those obtained under the natural Hecke action on the space of locally constant functions $f : D^*(A)/D^*_{\mathbf{Q}} \to \overline{\mathbf{F}}_p$. The result is proved by evaluating the modular forms at supersingular elliptic curves. In the letter to Kazhdan, Serre studies certain unramified representations of $\mathbf{GL}_2(\mathbf{Q}_\ell)$, in characteristic $p \neq \ell$, which are universal with the property of containing an eigenvector of the Hecke operator T_ℓ with a given eigenvalue a_ℓ. In an appendix to the paper, R. Livné mentions further developments of these questions. For example, a general study of the representations of $\mathbf{GL}_2(\mathbf{Q}_\ell)$ in characteristic $p \neq \ell$ was done later by Marie-France Vignéras (1989); the case $p = \ell$ has recently been studied by several people.

13.3.10. The general strategy of the work of Wiles (1995) and Taylor–Wiles (1995) on modular elliptic curves and Fermat's Last Theorem was presented by Serre at the Séminaire Bourbaki [S248, Œ 168(1995)]. The proof that any semistable elliptic curve defined over $\mathbf{Q}$ is modular is long and uses results of Ribet, Mazur, Langlands, Tunnell, Diamond, among others. On this occasion, Serre said that he did not claim to have verified all the technical details of the proof, "*qui sont essentiels, bien entendu*".

13.4. Abelian Varieties and ℓ-adic Representations. Let A be an abelian variety of dimension d defined over a field k. Given a prime $\ell \neq \mathrm{char}(k)$, the Tate module $T_\ell(A) = \varprojlim A[\ell^n](k_s)$ is a free $\mathbf{Z}_\ell$-module of rank $2d$. Let $V_\ell(A) = T_\ell(A) \otimes \mathbf{Q}_\ell$. The action of the absolute Galois group of k on the Tate module of A gives an ℓ-adic representation $\rho_\ell : \mathrm{Gal}(k_s/k) \to \mathbf{GL}(T_\ell(A)) = \mathbf{GL}_{2d}(\mathbf{Z}_\ell)$; its image, G_ℓ, is a compact subgroup of $\mathbf{GL}_{2d}(\mathbf{Z}_\ell)$, hence it is a Lie subgroup of the ℓ-adic Lie group $\mathbf{GL}(T_\ell)$. The Lie algebra $\mathfrak{g}_\ell$ of G_ℓ is a subalgebra of $\mathfrak{gl}(V_\ell)$ and does not change under finite extensions of the ground field; it acts on V_ℓ. When k is finitely generated over $\mathbf{Q}$, it is known that the rank of $\mathfrak{g}_\ell$ is independent of ℓ, but it is not known whether the same is true for the dimension of $\mathfrak{g}_\ell$. Serre has written several papers and letters on the properties of G_ℓ and $\mathfrak{g}_\ell$ (see below).

13.4.1. The first occurrence of the G_ℓ and the $\mathfrak{g}_\ell$ in Serre's papers can be found in [S109, Œ 62(1964)]; it was complemented a few years later by [S151, Œ 89(1971)]. When k is a number field, the Mordell–Weil theorem says that the group $A(k)$ of the k-rational points of A is a finitely generated abelian group. J.W.S. Cassels had asked whether it is true that every subgroup of finite index of $A(k)$ contains a congruence subgroup, at least when A is an elliptic curve. In the first paper, Serre transformed the problem into another one relative to the cohomology of the G_ℓ's, namely the vanishing of $H^1(\mathfrak{g}_\ell, V_\ell)$ for every ℓ and he solved it when $\dim(A) = 1$. In the second paper, he solved the general case by proving a vanishing criterion for the cohomology of Lie algebras which implies that $H^n(\mathfrak{g}_\ell, V_\ell) = 0$, for every n, and every ℓ.

13.4.2. In 1968, Serre and Tate published the seminal paper *Good reduction of abelian varieties* [S134, Œ 79(1968)]. Let K be a field, v a discrete valuation, O_v the valuation ring of v and k_v its residue field, which is assumed to be perfect. Given an abelian variety A defined over K, the authors start from the existence of the Néron model A_v of A with respect to v, which is a group scheme of finite type over Spec O_v. Serre and Tate define the concept of potential good reduction of A, which generalized that of good reduction. They prove that A has good reduction at v if and only if the Tate module $T_\ell(A)$ is unramified at v, where ℓ denotes a prime which differs from the characteristic of k_v. This criterion is partially due to A.P. Ogg (in the case of elliptic curves) and partially to I. Shafarevich. In their proof, the structure of the connected component $\widetilde{A}_v^0$ of the special fiber $\widetilde{A}_v$ of A_v appears: it is an extension of an abelian variety B by a linear group L, and L is a product of a torus S by a unipotent group U. The abelian variety A has good reduction if and only if $L = 1$; it has potential good reduction if and only if $L = U$; and it has semistable reduction if and only if $L = S$. A second theorem says that A has potential good reduction if and only if the image of the inertia group $I(\overline{v})$ for the ℓ-adic representation $\rho_\ell : \mathrm{Gal}(K_s/K) \to \mathrm{Aut}(T_\ell)$ is finite. An appropriate use of the characters of Artin and Swan then allows the definition of the conductor of A. The semistable reduction theorem, conjectured by Serre in 1964 and proved later by Grothendieck in (SGA 7), would allow the definition of the conductor for every abelian variety. (The semistable reduction theorem was also proved by D. Mumford, except that his proof, based on the use of theta functions, did not include the case where the residue characteristic is equal to 2.)

Suppose that A has complex multiplication by F over the field K, where F denotes an algebraic number field of degree $2d$, $d = \dim(A)$. In the same work, Serre and Tate prove that every abelian variety, defined over an algebraic number field K and with complex multiplication over this field, has potential good reduction at all the places of K, and that it has good reduction at the places of K outside the support of its Grössencharakter. This result generalizes some earlier ones of M. Deuring (1955) in the case of elliptic curves. The exponent of the conductor at v is given by $2dn_v$, where n_v is the smallest integer such that the Grössencharakter is zero when restricted to the ramification group $I(\overline{v})^{n_v}$, in the upper numbering.

13.4.3. In [S209, Œ 135(1985)], Serre explains how the theorems obtained by G. Faltings (1983) in his paper on the proof of Mordell's conjecture allow a better understanding of the properties of the ℓ-adic representations associated to abelian varieties.

In the first part of the lectures, Serre gives an effective criterion for showing that two ℓ-adic representations are isomorphic (the "*méthode des corps quartiques*"). This criterion was applied to prove that two elliptic curves, studied by J.-F. Mestre, of conductor 5077, are isogenous.

Let K be an algebraic number field and A an abelian variety defined over K of dimension d. Let $\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathbf{GL}(T_\ell(A))$ be the ℓ-adic representation defined by the Tate module. Let G_ℓ^{alg} be the closure of G_ℓ under the Zariski topology, which is a $\mathbf{Q}_\ell$-algebraic subgroup of the general linear group $\mathbf{GL}_{2d}$. Mumford and Tate conjectured that, given A and K, the group G_ℓ^{alg} is essentially independent of ℓ and, more precisely, that the connected component $(G_\ell^{\mathrm{alg}})^0$ could be deduced from the Mumford–Tate group by extension of scalars of $\mathbf{Q}$ to $\mathbf{Q}_\ell$. In the second part of the course [S209, Œ 135(1985)], Serre proves a series of results in this direction. He shows for instance that the finite group $G_\ell^{\mathrm{alg}}/(G_\ell^{\mathrm{alg}})^0$ is independent of ℓ.

13.4.4. In the course [S213, Œ 136(1986)], Serre studies the variation with ℓ of the ℓ-adic Lie groups associated to abelian varieties. Let us keep the previous notation. Given the homomorphism

$$\rho : \mathrm{Gal}(\overline{K}/K) \to \prod_\ell G_\ell \subset \prod_\ell \mathrm{Aut}(T_\ell),$$

Serre proves that, if K is sufficiently large, the image of ρ is open in the product $\prod_\ell G_\ell$, i.e. the ρ_ℓ are "almost independent". In the case where n is odd, or is equal to 2 or 6, and if $\mathrm{End}(A) = \mathbf{Z}$, he shows that the image of ρ is open in the product of the groups of symplectic similitudes $\prod \mathbf{GSp}(T_\ell, e_\ell)$. Here e_ℓ is the alternating form over $T_\ell(A)$ deduced from a polarization e of A. The ingredients of the proof are many: the above theorems of Faltings, Frobenius tori, McGill theory, properties of inertia groups at the places which divide ℓ, as well as group-theoretic information regarding the subgroups of $\mathbf{GL}_N(\mathbf{F}_\ell)$ supplied by theorems of V. Nori (1985–1987).

The proofs of the above results have not been published in a formal way, but one can find an account of them in Serre's letters to K. Ribet [Œ 133(1981)] and [Œ 138(1986)], to D. Bertrand [Œ 134(1984)], and to M.-F. Vignéras [Œ 137(1986)].

13.5. Motives. A first lecture on zeta and L-functions in the setting of the theory of schemes (of finite type over Spec($\mathbf{Z}$)) was given by Serre in [S112, Œ 64(1965)]. One finds in it a generalization of Chebotarev's density theorem to schemes of arbitrary dimension.

13.5.1. In his lecture in the Séminaire Delange–Pisot–Poitou [Œ 87(1969/70)], Serre introduces several definitions and formulates several conjectures about the local factors (gamma factors included) of the zeta function of a smooth projective variety over a number field. The local factors at the primes of good reduction do not raise any problem. The interesting cases are: (a) the primes with bad reduction; (b) the archimedean primes. In both cases Serre gives definitions based, in case (a), on the action of the local Galois group on the ℓ-adic cohomology, and in case (b), on the Hodge type of the real cohomology. The main conjecture is that such a zeta function has an analytic continuation to the s-plane and a very simple functional equation.

13.5.2. The subject of ℓ-adic representations had already been considered by Serre in [S177, Œ 112(1977)], in his address to the Kyoto Symposium on Algebraic Number Theory. In this paper, which is rich in problems and conjectures, we find the statement of the conjecture of Shimura–Taniyama–Weil, according to which any elliptic curve over $\mathbf{Q}$ of conductor N is a quotient of the modular curve $X_0(N)$.

13.5.3. The paper [S229, Œ 154(1991)] is a short introduction to the theory of motives. Along these lines, we also highlight the paper [S239, Œ 160(1993)], which corresponds to a text Serre wrote for Bourbaki in 1968. The paper deals with algebraic envelopes of linear groups and their relationship with different types of algebras, coalgebras and bialgebras. Its last section contains an account of the dictionary between compact real Lie groups and complex reductive algebraic groups.

13.5.4. To finish this section we shall briefly summarize the paper entitled *Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques* [S243, Œ 161(1994)]. Serre formulates a series of conjectures regarding ℓ-adic representations which generalize many of his previous results. We denote by M the category of pure motives over a subfield k of $\mathbf{C}$, which we suppose to be of finite type over $\mathbf{Q}$. The motivic Galois group G_M is related to the absolute Galois group of k by means of an exact sequence $1 \to G_M^0 \to G_M \to \mathrm{Gal}(\overline{k}/k) \to 1$. Given a motive E over k, let $M(E)$ be the smallest Tannakian subcategory of M which contains E. Suppose that the standard conjectures and Hodge conjecture are true. Under these assumptions and in an optimistic vein, Serre formulates a series of conjectures and questions aimed at the description of Grothendieck's "motivic paradise". We stress the following ones:

($1_?$) The motivic Galois group G_M is pro-reductive.
($2_?$) The motivic Galois group $G_{M(E)}$ is characterized by its tensor invariants.

(3?) The group $G_{M(E)/\mathbf{Q}_\ell}$ is the closure in the Zariski topology of the image of the ℓ-adic representation $\rho_{\ell,E} : \mathrm{Gal}(\bar{k}/k) \to G_{M(E)}(\mathbf{Q}_\ell)$, associated to E.

(4?) The connected pro-reductive group G_M^0 decomposes as $G_M^0 = C \cdot D$, where C is a pro-torus, equal to the identity component of the centre of G_M^0, and D is a pro-semisimple group, equal to the derived group of G_M^0.

(5?) If $S = (G_M^0)^{\mathrm{ab}}$, then S is the projective limit of the tori $T_{\mathfrak{m}}$ defined in his McGill book.

(6?) Every homomorphism $G_M^0 \to \mathbf{PGL}_2$ has a lifting to $G_M^0 \to \mathbf{GL}_2$.

(7) Which connected reductive groups are realized as $G_{M(E)}$? Are G_2 and E_8 possible?

(8?) The group $G_{\ell,E} = \mathrm{Im}(\rho_{\ell,E})$ is open in $G_{M(E)}(\mathbf{Q}_\ell)$. Let

$$\rho_E = (\rho_{\ell,E}) : \mathrm{Gal}(\bar{k}/k) \to \prod_\ell G_{\ell,E} \subset G_{M(E)}(\mathbf{A}^f).$$

Suppose that $G_{M(E)}$ is connected. Then E is a "maximal motive" if and only if $\mathrm{Im}(\rho_E)$ is open in the group $G_{M(E)}(\mathbf{A}^f)$, where $\mathbf{A}^f$ is the ring of the finite adeles of $\mathbf{Q}$.

The paper ends with a statement of the Sato–Tate conjecture for arbitrary motives.

14 Group Theory

In response to a question raised by Olga Taussky (1937) on class field towers, Serre proves in [S145, Œ 85(1970)] that for a finite p-group G, the knowledge of G^{ab} does not in general imply the triviality of any term $D^n(G)$ of its derived series. More precisely, for every $n \geqslant 1$ and for every non-cyclic finite abelian p-group P of order $\neq 4$, there exists a finite p-group G such that $D^n(G) \neq 1$ and $G^{ab} \simeq P$.

14.1. Representation Theory. Serre's popular book *Représentations linéaires des groupes finis* [S130, RLGF(1967)] gives a reader-friendly introduction to representation theory. It also contains less elementary chapters on Brauer's theory of modular representations, explained in terms of Grothendieck K-groups, the highlight being the "cde triangle". The text is well known to physicists and chemists[1], and its first chapters are a standard reference in undergraduate or graduate courses on the subject.

14.1.1. The paper [S245, Œ 164(1994)] is about the semisimplicity of the tensor product of group representations. A theorem of Chevalley (1954) states that, if k is a field of characteristic zero, G is a group, and V_1 and V_2 are two semisimple

[1]Indeed the first part of the book was written by Serre for the use of his wife Josiane who was a quantum chemist and needed character theory in her work.

$k(G)$-modules of finite dimension, then their tensor product $V_1 \otimes V_2$ is a semisimple $k[G]$-module. Serre proves that this statement remains true in characteristic $p > 0$, provided that p is large enough. More precisely, if V_i, $1 \leqslant i \leqslant m$, are semisimple $k[G]$-modules and $p > \sum(\dim V_i - 1)$, then the $k[G]$-module $V_1 \otimes \cdots \otimes V_m$ is also semisimple. The bound on p is best possible, as the case $G = \mathbf{SL}_2(k)$ shows. In order to prove this, Serre first considers the case in which G is the group of points of a simply connected quasi-simple algebraic group, and the representations V_1 and V_2 are algebraic, irreducible, and of restricted type. In this case, the proof relies on arguments on dominant weights due to J.C. Jantzen (1993). The general case is reduced to the previous one by using a "saturation process" due to V. Nori, which Serre had already used in his study of the ℓ-adic representations associated with abelian varieties (see Sect. 13.4 above). The study of these topics is continued in [S252, Œ 171(1997)], where one finds converse theorems such as: if $V \otimes V'$ is semisimple and $\dim(V')$ is not divisible by $\mathrm{char}(k)$, then V is semisimple. Here the proofs use only linear (or multilinear) algebra; they are valid in any tensor category.

14.1.2. In the Bourbaki report [S273 (2004), SEM(2008)], Serre extends the notion of complete reducibility (that is, semisimplicity) to subgroups Γ not only of $\mathbf{GL}_n$ but of any reductive group G over a field k. The main idea is to use the Tits building T of G. A subgroup Γ of G is called completely reducible in G if, for every maximal parabolic subgroup P of G containing Γ, there exists a maximal parabolic subgroup P' of G opposite to P which contains Γ. There is a corresponding notion of "G-irreducibility": Γ is called G-irreducible if it is not contained in any proper parabolic subgroup of G, i.e. if it does not fix any point of the building X. There is also a notion of "G-indecomposability". These different notions behave very much like the classical ones, i.e. those relative to $G = \mathbf{GL}_n$; for instance, there is an analogue of the Jordan–Hölder theorem and also of the Krull–Schmidt theorem. The proofs are based on Tits' geometric theory of spherical buildings. As one of the concrete applications given in the paper, we only mention the following: if $\Gamma \subset G(k)$, G is of type E_8 and V_i, $1 \leqslant i \leqslant 8$, denote the 8 fundamental irreducible representations of G, and if one of them is a Γ-module semisimple, then all the others are also semisimple provided that $\mathrm{char}(k) > 270$.

14.1.3. The Oberwolfach report [S272 (2004)], states without proof two new results on the characters of compact Lie groups. The first one is a generalization of a theorem of Burnside for finite groups: given an irreducible complex character χ of a compact Lie group G, of degree > 1, there exists an element $x \in G$ of finite order with $\chi(x) = 0$. The second one states that $\mathrm{Tr}(\mathrm{Ad}(g)) \geqslant -\mathrm{rank}(G)$ for all $g \in G$, the bound being optimal if and only if there is an element $c \in G$ such that $ctc^{-1} = t^{-1}$ for every $t \in T$, where T is a maximal torus of G; the proof is a case-by-case explicit computation (in the E_6 case, the computation was not made by Serre himself but by A. Connes).

14.1.4. In another Oberwolfach report [S279 (2006)], Serre defines the so-called Kac coordinates in such a way that they can be used to classify the finite subgroups of G which are isomorphic to μ_n, without having to assume that n is prime to the characteristic.

14.1.5. In 1974, Serre had asked W. Feit whether, given a linear representation

$$\rho : G \to \mathbf{GL}_n(K)$$

of a finite group G over a number field K, it could be realized over the ring of integers O_K. Although he did not expect a positive answer, he did not know of any counterexample. Given ρ, there are O_K-lattices which are stable under the action of G; but the point is that as O_K is a Dedekind ring, these lattices need not be free as O_K-modules. There is an invariant attached to them which lies in the ideal class group $C_K = \mathrm{Pic}(O_K)$ of K. Feit provided the following counterexample: if $G = Q_8$ is the quaternion group of order 8 and $K = \mathbf{Q}(\sqrt{-35})$, the answer to the question is no. The paper [S281 (2008)], reproduces three letters of Serre to Feit about this question, written in 1997. Their purpose was to clarify the mysterious role of $\sqrt{-35}$ in Feit's counterexample. Let $K = \mathbf{Q}(\sqrt{-N})$, for N a positive square free integer, $N \equiv 3 \pmod 8$. Let O_K denote its ring of integers. The field K splits the quaternion algebra $(-1,-1)$, hence there exists an irreducible representation V of degree 2 over K of Q_8. In the first letter, Serre proves that there exists an O_K-free lattice of V which is stable under the group Q_8 if and only if the integer N can be represented by the binary quadratic form $x^2 + 2y^2$. In order to prove this equivalence, Serre makes use of Gauss genus theory: any lattice $L \subset V$ stable under Q_8 yields an invariant $c(L)$ which lies in the genus group C_K/C_K^2 of the quadratic field K. It turns out that L is free as O_K-module if and only if $c(L) = 1$. The exact evaluation of the genus characters on $c(L)$ yields the criterion above.

Another version of the computation of the invariant $c(L)$ is explained in the second letter. Serre uses the fact, due to Gauss, that for a positive square-free integer N, $N \equiv 3 \pmod 8$, any representation of N as a sum of three squares yields an O_K-module of rank 1 which lies in a well defined genus and, moreover, every class in that genus is obtainable by a suitable representation of N as a sum of three squares. If $D = (-1,-1)$ denotes the standard quaternion algebra over $\mathbf{Q}$ and R is its Hurwitz maximal order, Serre embeds the ring of integers O_K in R by mapping $\sqrt{-N}$ to $ai + bj + ck$, where $a^2 + b^2 + c^2 = N$. It turns out that the invariant $c(R)$ is the same as the one obtained before. In the third letter, and more generally, given any quaternion algebra D over $\mathbf{Q}$ and an imaginary quadratic field K which splits D, if we choose an embedding $K \to D$, the O_K-invariant $c(O_D)$ of a maximal order O_D containing O_K does not depend on the choice of O_D. Serre determines $c(O_D) = c(D, K) \in C_K/C_K^2$ in terms of D and K. By making use of the Hilbert symbol, the genus group C_K/C_K^2 can be embedded in the 2-component of the Brauer group $\mathrm{Br}_2(\mathbf{Q})$; moreover the image of $c(D, K)$ in $\mathrm{Br}_2(\mathbf{Q})$ is equal to $(D) + (d_D, -d)$. In this formula, (D) denotes the element of the Brauer group defined by the quaternion algebra D, d_D is the signed discriminant of D, $-d$, with $d > 0$, is the discriminant of K, and $(d_D, -d)$ stands for the Hilbert symbol. In the special case $D = (-1,-1)$ and Q_8, the formula tells us that there exists a free

O_K-module of rank 2 which gives the standard irreducible representation of Q_8 over $K=\mathbf{Q}(\sqrt{-d})$ if and only if either $(-2,d)=0$ or $(-1,d)=0$; that is, if and only if d is representable either by x^2+2y^2 or by x^2+y^2. For example, if $d=8p$, $p\equiv 3 \pmod 8$, p prime, non-free lattices exist.

14.2. Algebraic Groups. The lecture course [S141, Œ 84(1969)] focused on discrete groups. Some of its contents would be published in [S139, Œ 83(1969)] and [S149, Œ 88(1971)]. Another part was published in the book *Arbres, amalgames, SL_2*, [S176, AA(1977)], written with the help of H. Bass. In the first chapter Serre shows that it is possible to recover a group G which acts on a tree X from the quotient graph or fundamental domain $G\backslash X$, and the stabilizers of the vertices and of the edges. If $G\backslash X$ is a segment, then G may be identified with an amalgam of two groups and, moreover, every amalgam of two groups can be obtained in this way. The study of relations between amalgams and fixed points show that groups such as $\mathbf{SL}_3(\mathbf{Z})$ and $\mathbf{Sp}_4(\mathbf{Z})$ are not amalgams, since one can show that they always have fixed points when they act on trees, see [S163 (1974)]; the method extends to all $G(\mathbf{Z})$, where G is any reductive group-scheme over $\mathbf{Z}$ which is simple of rank $\geqslant 2$.

In the second chapter, the results are applied to the study of the groups $\mathbf{SL}_2(k)$, where k is a local field. The group $\mathbf{SL}_2(k)$ acts on the Bruhat–Tits tree associated to the space k^2. The vertices of this tree are the classes of lattices of k^2. In this way, Serre recovers a theorem due to Y. Ihara by which every torsion-free discrete subgroup of $\mathbf{SL}_2(\mathbf{Q}_p)$ is free.

According to MathSciNet, this book is now Serre's most cited publication.

14.2.1. A question raised by Grothendieck concerning linear representations of group schemes was answered by Serre in [S136, Œ 81(1968)]. Suppose that C is a coalgebra over a Dedekind ring A which is flat. If Com_A denotes the abelian category of comodules over C which are of finite type as A-modules, one may consider the Grothendieck ring R_A of Com_A. Let K be the field of fractions of A. Serre proves that the natural morphism $i:\mathrm{R}_A\to\mathrm{R}_K$, $E\mapsto E\otimes K$, is an isomorphism if A is principal and under the assumption that all decomposition homomorphisms (defined as in Brauer's theory for finite groups) are surjective. If M is an abelian group and T_M denotes the A-group scheme whose character group is M, the bialgebra $C(M)$ can be identified with the group algebra $A[M]$. If A is principal, one has an isomorphism $ch:\mathrm{R}_A(T_M)\xrightarrow{\sim}\mathbf{Z}[M]$, provided by the rank. Next, Serre considers a split reductive group G and a split torus T of G, which exists by hypothesis. By composing ch with the restriction homomorphism $Res:\mathrm{R}_K(G)\to\mathrm{R}_K(T)$, a homomorphism $ch_G:\mathrm{R}_K(G)\to\mathbf{Z}[M]$ is obtained. Serre proves that ch_G is injective and that its image equals the subgroup $\mathbf{Z}[M]^W$ of the elements of $\mathbf{Z}[M]$ which are invariant under the Weyl group W of G relative to T. As an illustration of this result, the paper gives the following example: if $G=\mathbf{GL}_n$, $M=\mathbf{Z}^n$ and $W=S_n$ is the symmetric group on n elements, then $\mathbf{Z}[M]=\mathbf{Z}[X_1,\dots,X_n,X_1^{-1},\dots,X_n^{-1}]$ and $\mathrm{R}_A(\mathbf{GL}_n)=\mathrm{R}_K(\mathbf{GL}_n)=\mathbf{Z}[M]^W=\mathbf{Z}[\lambda_1,\dots,\lambda_n]_{\lambda_n}$, where $\lambda_1,\dots,\lambda_n$ denote the elementary symmetric functions in $X_1,\dots,X_n$, and the subscript stands for localization with respect to λ_n. This was what Grothendieck needed for his theory of λ-rings.

14.3. Finite Subgroups of Lie Groups and of Algebraic Groups. The problem of the determination of the finite subgroups of a Lie group has received a great amount of attention. Embedding questions of finite simple groups (and their non-split central extensions) in Lie groups of exceptional type have been solved by the work of many mathematicians. The paper [S250, Œ 167(1996)] contains embeddings of some of the groups $\mathbf{PSL}_2(\mathbf{F}_p)$ into simple Lie groups. Let G denote a semisimple connected linear algebraic group over an algebraically closed field k, which is simple of adjoint type; let h be its Coxeter number. The purpose of the paper is to prove that, if $p = h + 1$ is a prime, then the group $\mathbf{PGL}_2(\mathbf{F}_p)$ can be embedded into $G(k)$ (except if char$(k) \neq 2$ and $h = 2$), and that if $p = 2h + 1$ is a prime, then the group $\mathbf{PSL}_2(\mathbf{F}_p)$ can be embedded into $G(k)$. Since for $G = \mathbf{PGL}_2$ one has $h = 2$, the theorem generalizes the classical result that the groups $A_4 = \mathbf{PSL}_2(\mathbf{F}_3)$, $S_4 = \mathbf{PGL}_2(\mathbf{F}_3)$, and $A_5 = \mathbf{PSL}_2(\mathbf{F}_5)$ can be embedded into $\mathbf{PGL}_2(\mathbf{C})$. The result for $p = 2h + 1$ was known in the case of characteristic zero; it was part of a conjecture by B. Kostant (1983), and it had been verified case by case with the aid of computers. Moreover, the values $h + 1$ or $2h + 1$ for p are maximal in the sense that if $\mathbf{PGL}_2(\mathbf{F}_p)$, respectively $\mathbf{PSL}_2(\mathbf{F}_p)$, are embedded in $G(\mathbf{C})$ then $p \leqslant h + 1$, respectively $p \leqslant 2h + 1$.

In his paper, Serre proves the two results in a unified way.

One starts from a certain principal homomorphism $\mathbf{PGL}_2(\mathbf{F}_p) \to G(\mathbf{F}_p)$ if $p \geqslant h$. If $p = h + 1$, this homomorphism can be lifted to a homomorphism $\mathbf{PGL}_2(\mathbf{F}_p) \to G(\mathbf{Z}_p)$. A key point is that the Lie algebra L of $G_{/\mathbf{F}_p}$ turns out to be cohomologically trivial as a $\mathbf{PGL}_2(\mathbf{F}_p)$-module through the adjoint representation. This is not the case if $p = 2h + 1$, since then $H^2(\mathbf{PGL}_2(\mathbf{F}_p), L)$ has dimension 1; lifting to $\mathbf{Z}_p$ is not possible; one has to use a quadratic extension of $\mathbf{Z}_p$. Once this is done, the case where char$(k) = 0$ is settled. An argument based on the Bruhat–Tits theory gives the other cases.

As a corollary of the theorem, one obtains that $\mathbf{PGL}_2(\mathbf{F}_{19})$ and $\mathbf{PSL}_2(\mathbf{F}_{37})$ can be embedded in the adjoint group $E_7(\mathbf{C})$ and that $\mathbf{PGL}_2(\mathbf{F}_{31})$ and $\mathbf{PSL}_2(\mathbf{F}_{61})$ can be embedded in $E_8(\mathbf{C})$.

14.3.1. In his lecture [S260 (1999), SEM(2008)] delivered at the Séminaire Bourbaki (1998–1999), Serre describes the state of the art techniques in the classification of the finite subgroups of a connected reductive group G over an algebraically closed field k of characteristic zero. He begins by recalling several important results. For example, if p is a prime which does not divide the order of the Weyl group W of G, then every p-group A of G is contained in a torus of G and hence is abelian. The torsion set Tor(G) is, by definition, the set of prime numbers p for which there exists an abelian p-subgroup of G which cannot be embedded in any torus of G. The sets Tor(G), for G simply connected and quasi-simple, are well known; for instance Tor$(G) = \{2, 3, 5\}$ if G is of type E_8; moreover Tor$(G) = \emptyset$ is equivalent to $H^1(K, G) = 0$ for every extension K of k. For A a non-abelian finite simple group, Serre reproduces a table by Griess–Ryba (1999) giving the pairs (A, G) for which G is of exceptional type and A embeds projectively in G. In order to see that the table is, in fact, complete, the classification of finite simple groups is used.

14.3.2. Part of the material of the paper [S280 (2007)] arose from a series of three lectures at the École Polytechnique Fédérale de Lausanne in May 2005. Given a reductive group G over a field k and a prime ℓ different from char(k), and A a finite subgroup of $G(k)$, the purpose of the paper is to give an upper bound for $v_\ell(A)$, that is the ℓ-adic valuation of the order of A, in terms of invariants of G, k and ℓ. Serre provides two types of such bounds, which he calls S-bounds and M-bounds, in recognition of previous work by I. Schur (1905) and H. Minkowski (1887).

The Minkowski bound, $M(n, \ell)$, applies to the situation $G = \mathbf{GL}_n$ and $k = \mathbf{Q}$ and is optimal in the sense that for every n and for every ℓ there exists a finite ℓ-subgroup A of $\mathbf{GL}_n(\mathbf{Q})$ for which $v_\ell(A) = M(n, \ell)$. By making use of the (at the time) newly created theory of characters, due to Frobenius, Schur extended Minkowski's results to an arbitrary number field k: he defined a number $M_k(n, \ell)$ such that $v_\ell(A) \leqslant M_k(n, \ell)$ for any finite ℓ-subgroup of $\mathbf{GL}_2(\mathbf{C})$ such that $\mathrm{Tr}(g)$ belongs to k for any $g \in A$. As in the case $k = \mathbf{Q}$, Schur's bound is optimal. Both results were recalled by Serre in his lectures with almost complete proofs.

The S-bound for any reductive group and any finite subgroup of $G(k)$ is obtained in terms of $v_\ell(W)$, the ℓ-adic valuation of the order of the Weyl group of G and certain cyclotomic invariants of the field k, defined *ad hoc*. The Minkowski bound is more precise, but in order to obtain it, Serre needs to assume that the group G is semisimple of inner type (the action of $\mathrm{Gal}(k_s/k)$ on its Dynkin diagram is trivial). If r is its rank, then its Weyl group W has a natural linear representation of degree r. The ring of invariants $\mathbf{Q}[x_1, \dots, x_r]^W$ is a polynomial algebra $\mathbf{Q}[P_1, \dots, P_r]$, where P_i are homogenous polynomials of degrees $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r$. Under the assumption that G is semisimple of inner type, with root system R, Serre gives a Minkowski-style bound $M(\ell, k, R)$ for G which depends only on the ℓ-cyclotomic invariants of the field k and the degrees $d_i, i = 1, \dots, r$. Moreover, it is optimal, except when $\ell = 2$ and -1 does not belong to W. As an illustration, let us mention that, if G is a $\mathbf{Q}$-group of type E_8, then the order of any finite subgroup of $G(\mathbf{Q})$ divides

$$M(\mathbf{Q}, E_8) = 2^{30} \cdot 3^{13} \cdot 5^5 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 31,$$

and that this bound is sharp.

15 Miscellaneous Writings

Serre has written an endless number of impeccable letters over the years. They are now found as appendices of books, in papers or, simply, carefully saved in the drawers of mathematicians. As mentioned above, some of these letters are included in the *Œuvres*. Some were collected in the text edited by S.S. Chern and F. Hirzebruch in [Wolf Prize in Mathematics, vol. 2, World Scientific, 2001]: a letter to John

McCleary (1997), two letters to David Goss (1991, 2000), a letter to Pierre Deligne (1967), and a letter to Jacques Tits (1993). One finds in them comments on his thesis, on the writing of FAC, on ℓ-adic representations, as well as historical data on the "Shimura–Taniyama–Weil" modularity conjecture. In the letter to Tits, there is an account of the themes on which Serre was working in the years around 1993: Galois representations, inverse Galois problem, Abhyankar's problem, trace forms, and Galois cohomology. The Grothendieck–Serre correspondence [S263, GRSE(2001)], published more recently, is another invaluable resource for understanding the origins of the concepts and tools of current algebraic geometry.

15.0.1. Serre has written essays on the work of other mathematicians: for example, a publication of historical character on a prize delivered to J.S. Smith and H. Minkowski [S238 (1993)], or publications about the life and work of A. Weil [S259 (1999)] and that of A. Borel [S270 (2004)], [S271 (2004)]. He was the editor of the Collected Works of F.G. Frobenius [S135 (1968)], in three volumes. He and R. Remmert were the editors of the Collected Works of H. Cartan [S192 (1979)], also in three volumes. He was the editor of the Collected Works of R. Steinberg [S253 (1997)].

15.0.2. We should also mention expository papers that Serre likes to call "mathematical entertainment", where he takes a rather simple-looking fact as a starting point for explaining a variety of deeper results.

One such paper is [S206, Œ 140(1985)], whose title is just the high school discriminant formula $\Delta = b^2 - 4ac$. Given an integer Δ, one wants to classify the quadratic polynomials ax^2+bx+c with discriminant Δ, up to $\mathbf{SL}_2(\mathbf{Z})$-conjugation. This is a classical problem, started by Euler, Legendre and Gauss. Serre explains the results which were obtained in the late 1980s by combining Goldfeld's ideas (1976) with a theorem of Gross–Zagier (1986) and Mestre's proof (1985) of the modularity of a certain elliptic curve of conductor 5077 and rank 3.

Another such paper is [S268 (2002), SEM(2008)]. By an elementary theorem of C. Jordan (1872), if G is a group acting transitively on a finite set of $n > 1$ elements, the subset G_0 of the elements of G which act without fixed points is non-empty. Moreover, P.J. Cameron and A.M. Cohen (1992) have refined this result by proving that the ratio $|G_0|/|G| \geqslant 1/n$, and that it is $> 1/n$ if n is not a prime power. Serre gives two applications. The first one is topological and says that if $f : T \to S$ is a finite covering of a topological space S, of degree $n > 1$, and with T path-connected, then there exists a continuous map of the circle $\mathbf{S}_1$ in S which cannot be lifted to T. The second application is arithmetical and concerns the number of zeros $N_p(f)$ in the finite field $\mathbf{F}_p$ of a polynomial $f \in \mathbf{Z}[X]$. Serre shows that, if the degree of f is $n > 1$ and f is irreducible, then the set $P_0(f)$ of the primes p such that $N_p(f) > 0$ has a natural density $\geqslant 1/n$. The proof consists of combining Cameron–Cohen's theorem with Chebotarev's density theorem. The paper is illustrated with the computation of $N_p(f)$ for $f = x^n - x - 1$ and $n = 2, 3, 4$. In these three cases, it is shown how the numbers $N_p(f)$ can be read from the coefficients of suitable cusp forms of weight 1.

And, finally, let us mention the preprint *How to use finite fields for problems concerning infinite fields*, a mathematical entertainment just written by Serre [S284 (2009)] in which he discusses old results of P.A. Smith (1934), M. Lazard (1955) and A. Grothendieck (1966), and shows how to prove them (and sometimes improve them) either with elementary tools or with topological techniques.

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Degrees/education:
École Normale Supérieure (Paris), 1945–1948
Agrégé des sciences mathématiques, 1948
Docteur ès sciences, Sorbonne, 1951

Positions:
Attaché puis chargé de recherches, CNRS, 1948–1953
Maître de recherches, CNRS, 1953–1954
Maître de conférences, Faculté des Sciences de Nancy, 1954–1956
Professeur, Collège de France, 1956–1994
Professeur honoraire, Collège de France, 1994–

Visiting positions:
Harvard University (1957, 1964, 1974, 1976, 1979, 1981, 1983, 1985, 1988, 1990, 1992, 1994, 1995, 1996, 2003, 2005, 2007)
Institute for Advanced Study, Princeton (1955, 1957, 1959, 1961, 1963, 1967, 1970, 1972, 1978, 1983, 1999)
I.H.E.S., Bures-sur-Yvette (1963–1964)
Göttingen Universität (1970)
McGill University (1967, 2006)
Mexico University (1956)
Princeton University (1952, 1999)
Singapore University (1985)

Memberships:
American Academy of Arts and Sciences, 1960
Académie des Sciences de Paris (correspondant : 1973, titulaire : 1977)
London Mathematical Society (Honorary Member), 1973
Fellow of the Royal Society, 1974
Royal Netherlands Academy of Arts and Sciences, 1978
National Academy of Sciences (USA), 1979

Royal Swedish Academy of Sciences, 1981
American Philosophical Society, 1998
Russian Academy of Sciences, 2003
Norwegian Academy of Science and Letters, 2009

Awards and prizes:

Fields Medal, 1954
Prix Peccot–Vimont, 1955
Prix Francoeur, 1957
Prix Gaston Julia, 1970
Médaille Émile Picard, 1971
Balzan Prize, 1985
Médaille d'or du C.N.R.S., 1987
AMS Steele Prize, 1995
Wolf Prize, 2000
Abel Prize, 2003

Honorary degrees:

Cambridge, 1978
Stockholm, 1980
Glasgow, 1983
Athens, 1996
Harvard, 1998
Durham, 2000
London, 2001
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