2005

Peter D. Lax

ABEL

PRISEN

Autobiography

Peter D. Lax

Like most mathematicians, I became fascinated with mathematics early, about age ten. I was fortunate that my uncle could explain matters that puzzled me, such as why minus times minus is plus—it follows from the laws of algebra.

Mathematics had a deep tradition in Hungary, going back to the epoch-making invention of non-Euclidean geometry by János Bolyai, an Hungarian genius in the early 19th century. To this day, the Hungarian mathematical community seeks out mathematically talented students through contests and a journal for high school students. Winners are then nurtured intensively. I was tutored by Rose Peter, an outstanding logician and pedagogue; her popular book on mathematics, "Playing with Infinity" is still the best introduction to the subject for the general public.

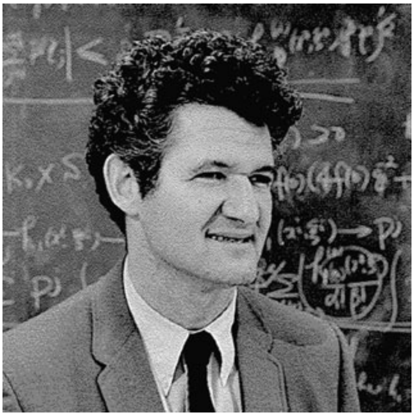
At the end of 1941 I came to the US with my family, sailing from Lisbon on December 5, 1941. It was the last boat to America; I was 15 years old. My mentors wrote to Hungarian mathematicians who had already settled in the US, asking them to take an interest in my education. They were very supportive.

I finished my secondary education at Stuyvesant High School, one of the elite public schools in New York City; its graduates include many distinguished mathematicians and physicists. For me the important thing was to learn English, and the rudiments of American history. In the meanwhile I visited from time to time Paul Erdős at the Institute for Advanced Study at Princeton. He was extremely kind and supportive; he would give me problems, some of which I managed to solve. My first publication, in 1944, was "Proof of a conjecture of P. Erdős on the derivative of a polynomial".

At the suggestion of Gabor Szegő I enrolled at New York University in the Spring of 1943 to study under the direction of Richard Courant, widely renowned for nurturing young talent. It was the best advice I ever received. But my studies came to a temporary halt in June 1944, when I was drafted into the US Army. I became an American citizen during my basic training.

P.D. Lax (✉)
Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA
e-mail: lax@courant.nyu.edu

H. Holden, R. Piene (eds.), *The Abel Prize*,
DOI 10.1007/978-3-642-01373-7_8,



(a)

(b)

(c)

I was then sent to Texas A & M to study engineering. There I passed the preliminary test in calculus with flying colors, and together with another soldier with similar background was excused from the course. Professor Klipple, a former student of R.L. Moore, generously offered to conduct just for the two of us a real variables course in the style of R.L. Moore. I learned a lot.

In June, 1945 I was posted to Los Alamos, the atomic bomb project. It was like living science fiction; upon arrival I was told that the whole town of 10,000 was engaged in an effort to build an atomic bomb out of plutonium, an element that does

(d)

With Lori Courant on the day I was notified about the Abel Prize

not exist in nature but was manufactured in a reactor at Hanford, WA. The project was led by some of the most charismatic leaders of science. I was assigned to the Theoretical Division; I joined a number of bright young soldiers, Dick Bellman, John Kemeny, Murray Peshkin and Sam Goldberg.

Von Neumann was a frequent consultant; each time he came, he gave a seminar talk on mathematics. He had little time to prepare these talks, but they were letter perfect. Only once did he get stuck; he excused himself by saying that he knew three ways of proving the theorem, but unfortunately chose a fourth.

Enrico Fermi and Niels Bohr were also consultants; since they were closely associated with nuclear physics, Security insisted that they use code names; Fermi became Henry Farmer, Bohr Nicholas Baker. At a party a woman who had spent some time in Copenhagen before the war recognised Bohr, and said "Professor Bohr, how nice to see you". Remembering what Security had drilled into him, Bohr said "No, I am Nicholas Baker", but then immediately added "You are Mrs Houtermans". "No", she replied, "I am Mrs Placzek". She had divorced and remarried in the meantime.

I returned to New York University in the Fall of 1946 to get my undergraduate degree, and simultaneously to continue my graduate studies. I joined an outstanding class of graduate students in mathematics, Avron Douglis, Eugene Isaacson, Joe Keller, Martin Kruskal, Cathleen Morawetz, Louis Nirenberg and Anneli Kahn.

I got my PhD in 1949 under the direction of K.O. Friedrichs, a wonderful mathematician and a delightful, idiosyncratic person. He kept his life on a strict schedule; he knew that this was absurd, but it worked for him. When a graduate student of his repeatedly delayed finishing his dissertation, I explained to Friedrichs that the student was very neurotic. That seemed to him no excuse; "Am I not just as neurotic?", he said, "yet I finish my work".

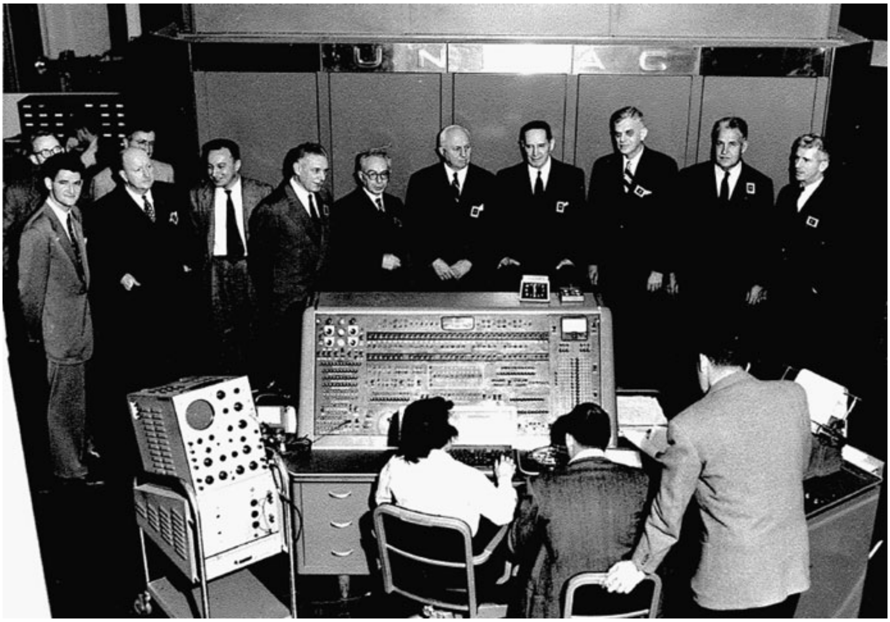
(e)

In the course of some research we did jointly, I found a reference in a Russian journal. Friedrichs said that he knew the alphabet, a couple of hundred words, and the rudiments of Russian grammar, and was willing to read the paper. I was worried about the language barrier, but Friedrichs said: "That it is in Russian is nothing; the difficulty is that it is mathematics".

Anneli and I married in 1948; our first child, John, was born in 1950. We spent part of 1950/51 at Los Alamos. We returned to New York in the fall of '51 to take up an appointment as Research Assistant Professor, in the mathematics department of New York University. I remained in the department for nearly fifty years, basking in the friendly collegial atmosphere of the place.

In the fifties I spent most of my summers at Los Alamos. At that time under the leadership of von Neumann, Los Alamos was the world leader in numerical computing and had the most up-to-date computers. I became, and remained, deeply involved in problems of the numerical solutions of hyperbolic equations, in particular the equations of compressible flow.

In 1954 the Atomic Energy Commission placed a Univac computer at the Courant Institute; it was the first supercomputer, with a thousand words of memory. Our first task was to calculate the flood stages on the Columbia river in case the Grand Cooley dam were destroyed by sabotage. The AEC wanted to know if the



(f) In front of the UNIVAC in 1954. From left: Peter Lax, Gen. Willoughby (adjutant of Gen. MacArthur), Lazer Bromberg (Director of Courant Computing Center), Gus Kinzel (Chairman of Courant Council), Richard Courant, James Rand (CEO of Remington-Rand), Gen. MacArthur, Henry Heald (President of NYU), Gen. Groves (Head of the Manhattan Project), Gen. Howley (Commander of the Berlin airlift, Vice President at NYU)

Hanford reactor would be flooded. Originally the Corps of Engineers was charged with this task, but it was beyond their capabilities. The team at the Courant Institute, led by Jim Stoker and Eugene Isaacson, found that the reactor would be safe.

The Univac was manufactured by the Remington–Rand Corporation. The official installation of the computer at New York University was a sufficiently important event for James Rand, the CEO, to attend and bring along some members of his Board of Directors, including the chairman, General Douglas McArthur, and General Leslie Groves. In his long career General McArthur had been Commandant of the Corps of Engineers; he was keenly interested in our calculations, and grasped the power of modern computers.

The postwar years were a heady time for mathematics, in particular for the theory of partial differential equations, one of the main lines of research at the Courant Institute as well as other institutes here and abroad. The subject is a wonderful mixture of applied and pure mathematics; most equations describe physical situations, but then take on a life of their own.

Richard Courant retired as Director of the Institute in 1958, at the age of 70. His successor was Jim Stoker who accomplished the crucial task of making the Institute part of New York University by securing tenure for its leading members.

After Stoker retired, the younger generation took over, Jürgen Moser, Louis Nirenberg, the undersigned, Raghu Varadhan, Henry McKean, Cathleen Morawetz, and others; our present Director is Leslie Greengard. Significant changes took place during these years, but the Institute adhered to the basic principle that had guided Richard Courant: not to pursue the mathematical fashion of the day ("I am against panic buying in an inflated market") but to hire promising young people. Also, for Courant mathematics was a cooperative enterprise, not competitive.

In the sixties Ralph Phillips and I embarked on a project to study scattering theory. Our cooperation lasted 30 years and led to many new results, including a reformulation of the theory. This reformulation was used by Ludvig Faddeev and Boris Pavlov to study automorphic functions; they found a connection between automorphic scattering and the Riemann hypothesis.

Also in the sixties Martin Kruskal and Norman Zabusky, guided by extensive numerical computations, found remarkable properties of solutions of the Korteweg–de Vries equation. These eventually led to the discovery that the KdV equation is completely integrable, followed by the discovery, totally unsuspected, of a whole slew of completely integrable systems. I had the pleasure and good luck to participate in this development.

In 1970 a mob protesting the Vietnam war invaded the Courant Institute and threatened to blow up our CDC computer. They left after 48 hectic hours; my colleagues and I in the Computing Center smelled smoke and rushed upstairs just in time to disconnect a burning fuse. It was a foolhardy thing to do, but we were too angry to think.

In 1980 I was appointed to a six year term on the National Science Board, the policy making body of the National Science Foundation. It was an immensely gratifying experience; I learned about issues in many parts of science, as well as about the politics of science. My colleagues were outstanding scientists and highly colorful characters.

By the time the eighties rolled around, the Government no longer placed supercomputers at universities, severely limiting the access of academic scientists to computing facilities. My position on the Science Board gave me a chance to remedy this intolerable situation. A panel I chaired recommended that the NSF set up regional Computing Centers, accessible to distant users through high capacity lines. The Arpanet Project, the precursor of the Internet, demonstrated the practicality of such an arrangement.

I have always enjoyed teaching at all levels, including introductory calculus. At the graduate level my favorite courses were linear algebra, functional analysis, and partial differential equations. The notes I have prepared while teaching formed the basis of the books I have written on these subjects.

I supervised the PhD dissertations of 55 graduate students; many have become outstanding mathematicians. Some became close personal friends.

I retired from teaching in 1999, shortly after reaching the age 70. According to a US law passed in 1994, nobody can be forced to retire on account of age in any profession, including teaching at a university. This sometimes had unwelcome consequences, such as the case of a professor at a West Coast university who stayed on the faculty well into his seventies. Eventually his colleagues petitioned the administration to retire him on the ground that he is a terrible teacher. At a hearing he had a chance to defend himself; his defense was, "I have always been a terrible teacher".

In retirement I occupy myself by writing books, and by continuing to puzzle over mathematical problems. I receive invitations to visit and lecture at mathematical centers. I attend the annual meeting of the American Mathematical Society. I spend a lot of time with my friends and my family, including three rapidly growing grandsons. Anneli died in 1999; Lori Courant and I were fortunate to find each other, and we are living happily ever after.

Mathematics is sometimes compared to music; I find a comparison with painting better. In painting there is a creative tension between depicting the shapes, colors and textures of natural objects, and making a beautiful pattern on a flat canvas. Similarly, in mathematics there is a creative tension between analyzing the laws of nature, and making beautiful logical patterns.

Mathematicians form a closely knit, world wide community. Even during the height of the Cold War, American and Soviet scientists had the most cordial relations with each other. This comradeship is one of the delights of mathematics, and should serve as an example for the rest of the world.

Added by the Editors: One can find an interview with Peter and Anneli Lax in *More Mathematical People* (D.J. Albers, G.L. Alexanderson, and C. Reid, eds.), Hartcourt Brace Jovanovich Publishers, Boston, 1990.

A Survey of Peter D. Lax's Contributions to Mathematics

Helge Holden and Peter Sarnak

1 Introduction

Peter D. Lax has given seminal contributions to several areas of mathematics. In this paper we have decided to organize our discussion according to his *Selected Papers*, edited by P. Sarnak and A. Majda, and published in two volumes by Springer in 2005 [L215]–[L216].[1] We have benefited from the comments given there. As it is impossible to cover his entire contributions to many areas of pure and applied mathematics, we have tried to make a selection of some of the highlights of a career that spans more than six decades. His research is marked by original and concise analysis, using elementary means whenever possible (but he is never shy of using

Research supported in part by the Research Council of Norway.

[1]References of the form [Ln] (n a natural number) refer to Lax's list of publications.

H. Holden (✉)
Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway
e-mail: holden@math.ntnu.no
url: http://www.math.ntnu.no/~holden/

H. Holden
Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, 0316 Oslo, Norway

P. Sarnak
School of Mathematics, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540, USA
e-mail: sarnak@Math.Princeton.edu
url: http://www.mat.univie.ac.at/~jmichor/

P. Sarnak
Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA

H. Holden, R. Piene (eds.), *The Abel Prize*,
DOI 10.1007/978-3-642-01373-7_9, © Springer-Verlag Berlin Heidelberg 2010

advanced techniques if required) to reveal new and fundamental relations. Thus his research never goes out of vogue.

Lax has been a key figure in the development of numerical methods for partial differential equations and scientific computing since its inception in the aftermath of World War II at Los Alamos to the present prolific use of computer simulations in all areas of science and technology. Lax has always stressed the interplay between mathematical analysis and numerical experiments, as a source of mutual inspiration. Indeed, Lax once remarked [1] that "Computer simulations play a big role [in mathematics]. After all, the great mathematician G.D. Birkhoff believed all his life that the ergodic hypothesis was true and devoted much of his life to studying it. If he had been able to take one look at a computer simulation, he would have seen that it wasn't so". In [L157] he elaborated further "It is impossible to exaggerate the extent to which modern applied mathematics has been shaped and fueled by the general availability of fast computers with large memories. Their impact on mathematics, both applied and pure, is comparable to the role of telescopes in astronomy and microscopes in biology".

From 1980 to 1986, Peter Lax chaired the National Science Board which was highly instrumental in making supercomputers available to university scientists while stressing the importance of further research in the area. The *Lax Report*, or as its official name reads, "Report of the Panel on Large Scale Computing in Science and Engineering", from 1982, under the sponsorship of the Department of Defense and the National Science Foundation (NSF), was very influential in stressing the importance of the enhanced use of supercomputers in science and engineering, and by necessity, increased research in computational mathematics. Furthermore, it led to the establishment of NSF's five national computing centers as well as NSFnet.

In addition to the research papers, Peter Lax has written a number of books, including a university calculus book [L132], a book on linear algebra [L182] (two editions), a book on functional analysis [L205], two books on scattering theory [L53] (two editions), [L94], two books on hyperbolic differential equations [L83, L222], and lecture notes on partial differential equations [L5]. Furthermore, he has shown his exceptional ability as an expositor in a number of survey papers, both technical and nontechnical. For the paper [L46] on numerical solutions of partial differential equations, he received the Lester R. Ford Award of the Mathematical Association of America, and for the survey paper [L79] on shock waves, he received the Chauvenet Prize as well as the Lester R. Ford Award, both of the Mathematical Association of America. He has written several papers where he revisits old theorems from a new angle and always with an elegant twist, the latest being a new proof of Cauchy's integral theorem [L225]. In addition, he has written a number of portraits and obituaries of fellow scientists, showing a deep sense of human values, and a gift for exposition. We refer to the list of publications.

We end this introduction with a brief discussion of Peter's first paper [L1], written when he was 17 years old, and published in the country where he had newly arrived, and where he would remain for the rest of his career. In the paper he resolves a conjecture by his compatriot Erdős. If P is a polynomial of degree n, Bernstein's inequality asserts that

$$\max_{|z|\le 1} |P'(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The inequality turns into an equality if and only if $P(z) = az^n$. Erdős conjectured that if P had no zeros in $|z| < 1$, then the n could be replaced by $n/2$, with equality for $P(z) = (z^n + 1)/2$. In his first paper, Lax elegantly proved this conjecture.

It is perhaps fitting to end this introduction with Peter Lax's advice to the young generation [L157] "I heartily recommend that all young mathematicians try their skill in some branch of applied mathematics. It is a gold mine of deep problems whose solutions await conceptual as well as technical breakthroughs. It displays an enormous variety, to suit every style; it gives mathematicians a chance to be part of the larger scientific and technological enterprise. Good hunting!"

2 Partial Differential Equations—General Results

A substantial part of Lax's work has been in partial differential equations. In this section we collect some of his more general results. More specific results, concerning difference approximations, hyperbolic equations, and integrable systems, are treated in separate sections.

The *Lax–Milgram theorem* was proved in [L11].[2] Although only a slight extension of the Riesz representation theorem and easily proved, it has turned out to be exceptionally useful, and it is now a household theorem in textbooks. In a slightly more general version (cf. [p. 57, L205]) it reads:

Theorem 2.1 *Let H be a Hilbert space. Assume that $B\colon H \times H \to \mathbb{C}$ satisfies:*

(i) *$B(x, y)$ is linear in x for each fixed y, and $B(x, y)$ is skew linear in y for each fixed x.*
(ii) *B is bounded, i.e., $|B(x, y)| \le C_1 \|x\| \, \|y\|$ for some constant C_1.*
(iii) *B is bounded from below, i.e., $|B(x, x)| \ge C_2 \|x\|^2$ for some positive constant C_2.*

Then the following assertion holds: For any bounded linear functional ℓ on H there exists a unique $y \in H$ such that

$$\ell(x) = B(x, y), \quad x \in H.$$

[2] Lax describes the background as follows [p. 116, L215]: "Arthur Milgram was an excellent topologist at the University of Minnesota. . . . We became friends, and he asked me for a problem to work on. I explained to him how variational arguments can be used to extend self-adjoint operators that are bounded from below, but that there is no known method for dealing with operators that are not symmetric. After some thought he came up with this theorem".

As an example of an application of the Lax–Milgram theorem we mention the following [26, Sect. 6.2]: Define the operator

$$B(u,v)=\int_\Omega\left(\sum_{j,k=1}^n a^{jk}u_{x_j}v_{x_k}+\sum_{j=1}^n b^j u_{x_j}v+cuv\right)dx,\quad u,v\in H_0^1(\Omega),\tag{2.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^n$, and a^{jk}, b^j, c are bounded functions. Define L by

$$Lu=-\sum_{j,k=1}^n\left(a^{jk}u_{x_j}\right)_{x_k}+\sum_{j=1}^n b^j u_{x_j}+cu.\tag{2.2}$$

Then it follows from the Lax–Milgram theorem that there exists a number $\gamma\geq 0$ such that for each $\mu\geq\gamma$ and each $f\in L^2(\Omega)$ there exists a unique weak solution $u\in H_0^1(\Omega)$ of the boundary-value problem

$$Lu+\gamma u=f \text{ in } \Omega,\quad u=0 \text{ on } \partial\Omega.\tag{2.3}$$

In the short paper [L18], Lax and Courant considered the propagation of singularities in hyperbolic systems, and they showed that discontinuities in the initial data on a spacelike manifold are propagated along characteristics. The problem was studied from a more general point of view by Lax in the pioneering paper [L23], and a major step forward was taken by his analysis of asymptotic solutions of oscillatory initial value problems for first-order hyperbolic equations. The paper represents the genesis of the theory of Fourier integral operators (see [38, 39]), microlocal analysis and semiclassical limits. In typical fashion, Lax's analysis in [L23] is novel and crisp, and uses elementary methods, unmistakably his fingerprint. To be more precise, he considers first-order hyperbolic systems on $\mathbb{R}^m\times\mathbb{R}$ given by

$$Mu=U_t+\sum_{j=1}^m A_j u_{x_j}+Bu,\quad u=u(x,t)\in\mathbb{R}^n,\tag{2.4}$$

where A_j and B are $n\times n$ matrices with entries that are C^∞ functions of $(x,t)\in\mathbb{R}^m\times\mathbb{R}$. It is assumed that in a neighborhood of the hyperplane $t=0$ the operator M is space-like in the sense that all eigenvalues of the matrix

$$\mathcal{M}(p_1,\dots,p_m)=\sum_{j=1}^m p_j A_j,\tag{2.5}$$

are real and distinct for all $p_j\in\mathbb{R}$. Lax assumes that the initial data are given as

$$u|_{t=0}(x,0)=e^{i\xi\ell(x,0)}\phi(x),\tag{2.6}$$

and he posits that the solution of $Mu = 0$ takes the form

$$u(x,t) \sim e^{i\xi\ell(x,t)} \sum_{k=0}^{\infty} v_k(x,t)\xi^{-k}. \tag{2.7}$$

The natural next step is to insert the ansatz (2.7) into the equation $Mu = 0$, and solve recursively for powers of ξ^{-k}. That gives the well-known eikonal equation of geometric optics. If $\lambda = \lambda(x,t,p_1,\ldots,p_m)$ denotes an eigenvalue for the matrix $\mathcal{M}$, we obtain the first-order nonlinear scalar equation

$$\lambda_t = \lambda(x,t,\nabla_x \ell), \tag{2.8}$$

which can be solved for small t by the standard method of characteristics, which are the bi-characteristics of the original equation. The functions v_k are determined by solving the corresponding transport equations. Next he expands the δ function as in (2.6), a result that is used to construct a fundamental solution of (2.4) up to a smoothing operator. More precisely, for any natural number n, Lax provides a distributional kernel $K_n(t,x,y)$ which differs from the fundamental solution $K(t,x,y)$ by a function that is C^{n+1} in all variables. The fundamental solution is used to derive properties regarding the propagation of singularities for (2.4). This theory was further developed 15 years later by, e.g., Hörmander and Duistermaat (see, e.g., [23, 39]).

To show the power of Lax's method one may consider the following example: Let X be a smooth compact Riemannian manifold of dimension ν, and let Δ be the Laplace–Beltrami operator on X. We can then find an orthonormal basis of eigenfunctions ϕ_j with corresponding eigenvalues k_j^2, thus,

$$-\Delta\phi_j = k_j^2\phi_j. \tag{2.9}$$

Consider next the hyperbolic wave equation

$$u_{tt} = \Delta u, \qquad u(x,0) = \delta(x,y), \quad u_t(x,0) = 0 \tag{2.10}$$

on $\mathbb{R} \times X$. The fundamental solution equals

$$K(t,x,y) = \sum_{j=0}^{\infty} \cos(k_j t)\phi_j(x)\phi_j(y). \tag{2.11}$$

The Lax construction yields the singular part of $K(t,x,y)$ for small t, which allows for investigation of asymptotic behavior of sums like

$$\sum_{\lambda \le k_j \le \lambda+1} \phi_j(x)\phi_j(y) \text{ as } \lambda \to \infty. \tag{2.12}$$

Applications include, e.g., Weyl's law with remainder [39]. More precisely,

$$N(\lambda) = \sum_{k_j \le \lambda} 1 = (2\pi)^{-\nu} c_\nu \mathrm{Vol}(X)\lambda^\nu + \mathcal{O}(\lambda^{\nu-1}). \tag{2.13}$$

These results are sharp for a ν-dimensional sphere with its standard metric. However, if X has negative curvature, the results are not sharp. One problem is to understand (2.12) with shorter sums where $\lambda \le k_j \le \lambda + 1$ is replaced by $\lambda \le k_j \le \lambda + \eta(\lambda)$ with $\eta(\lambda) \sim \lambda^{-\alpha}$ for some α positive. This is still a major challenge in the area. See [24] and the more recent [68].

It is by now well-known that quasilinear systems of hyperbolic equations develop singularities in finite time, even for smooth initial data,[3] but in the early days of the theory, rigorous general results were absent. While it is fairly easy to see in the scalar case, it is considerably more difficult to establish this in the case of systems. In [L41] Lax applies a simple argument that provides the first rigorous proof for breakdown of solution, by analyzing the Riemann invariants in the genuinely nonlinear case (see Sect. 4), in the case of a 2×2 system. A similar result was also obtained by Oleĭnik [55]. Klainerman and Majda extended the work by Lax to the linearly degenerate case [44].

In a joint paper with Nirenberg [L48], Lax studies Gårding's inequality in the context of difference operators in order to show stability of difference schemes. The sharp Gårding inequality reads as follows. Consider a differential operator $A = a(x, D)$ where $a(x, \xi)$ is an $n \times n$ Hermitian matrix whose elements are polynomials in ξ. If $a(x, \xi)$ is homogeneous in ξ of degree r, smooth, and positive definite, then the sharp Gårding inequality states that there exists a constant K such that

$$\mathrm{Re}(Au, u) \ge -K\|u\|^2_{(r-1)/2}. \tag{2.14}$$

In [L48] Lax and Nirenberg provide a new proof in the matrix case, extending the proof by Hörmander [37] in the scalar case. The discrete version considers difference operators of the form

$$P_\delta = \sum_\alpha p_\alpha(x) T^\alpha \tag{2.15}$$

where α is a multi-index, T^α denotes the shift operator $(T^\alpha u)(x) = u(x + \delta\alpha)$, and $p_\alpha(x)$ are $m \times m$ matrix functions of x. If the symbol $p(x, \xi)$ of P_δ is sufficiently smooth, and is an Hermitian and nonnegative matrix for every (x, ξ), i.e., $p(x, \xi) \ge 0$, then

$$\mathrm{Re}\, P_\delta \ge -K\delta \tag{2.16}$$

for some constant K. The result is used to infer the stability of a class of difference schemes.

Peter Lax's name is associated with several other quantities, that we for reasons of brevity are unable to discuss in detail. We mention the following: The frequently used term *negative Lax norm* originated in the paper [L16], and the *Lax–Mizohata theorem* developed from [L23]. The so-called *Lax conjecture* originated in [L24].

[3] This is extensively discussed in Sect. 4.

3 Difference Approximations to Partial Differential Equations

The *Lax* or *Lax–Richtmyer stability theorem* appeared in [L17] in 1956. Those were the early days of computer simulations of difference schemes for partial differential equations. The main question was (as it still is): Does the numerical scheme converge to the solution? The quest for the true solution involves the application of finer and finer resolution to improve the approximation. Is the computed approximation stable? In setting up the scheme, a first question is whether the scheme is consistent with the underlying differential equation, i.e., does it approximate the equation? Three notions are thus involved: Consistency, stability, and convergence. The Lax stability theorem says that for linear initial value problems, stability is necessary and sufficient for convergence if the scheme is consistent. More precisely, we can formulate the theorem as follows: Let A be a linear operator (involving spatial derivation, matrix multiplications, etc.) and consider the initial value problem

$$u_t = Au(t), \quad u|_{t=0} = u_0. \tag{3.1}$$

By a genuine solution of (3.1) we mean a one-parameter family $u(t)$ such that

$$\left\| \frac{1}{\tau}(u(t+\tau) - u(t)) - Au(t) \right\| \underset{\tau\to 0}{\longrightarrow} 0, \quad \text{uniformly in } t \text{ for } t \in [0, T]. \tag{3.2}$$

Assume that the problem is properly posed in the sense that there exists a uniformly bounded semigroup $E_0(t)$ such that $u(t) = E_0(t)u_0$ is the solution, thus

$$\|E_0(t)u_0\| \le K\|u_0\|, \quad t \in [0, T]. \tag{3.3}$$

If E_0 is defined on a dense subset of some space $\mathcal{B}$, we can extend E_0 to some bounded and linear extension E that satisfies the same bound (3.3). Introduce now a finite difference approximation. To that end let Δt be the time discretization parameter, and define $t_n = n\Delta t$ for $n \in \mathbb{N}$. Write u^n for the approximation to $u(t_n)$, that is, $u^n \approx u(t_n)$. Derivatives are replaced by finite differences, e.g., $u_t(t) \approx (u(t + \Delta t) - u(t))/\Delta t$, and similar for spatial derivatives. This turns the differential equation into a discrete equation where the unknown function is evaluated on a lattice in space and time.[4] A finite difference scheme is then a recipe that describes how to compute the lattice-valued approximate solution. In this framework the scheme is encoded in an operator $C(\Delta t)$ such that

$$u^{n+1} = C(\Delta t)u^n. \tag{3.4}$$

We say that $C(\Delta t)$ is consistent (with A) if

$$\lim_{\Delta t\to 0} \left\| \left(\frac{1}{\Delta t}\big(C(\Delta t) - I\big) - A \right) u(t) \right\| = 0, \quad \text{uniformly in } t \text{ for } t \in [0, T]. \tag{3.5}$$

[4] An explicit example is given in (4.9).

We say that $C(\Delta t)$ converges to A if

$$\left\| C(\Delta t)^n u_0 - E(t)u_0 \right\| \underset{\Delta t\to 0,\ n\Delta t = t}{\longrightarrow} 0, \quad t \in [0, T]. \tag{3.6}$$

Finally, we say that $C(\Delta t)$ is stable if $C(\Delta t)^n$ remains uniformly bounded for $n\Delta t \in [0, T]$ and all $\Delta t \le \tau$ for some fixed τ. We have all the results we need to state the Lax–Richtmyer stability theorem.

Theorem 3.1 *Given the properly posed initial value problem* (3.1), *and a finite difference approximation $C(\Delta t)$ to it that satisfies the consistency condition, stability is a necessary and sufficient condition that $C(\Delta t)$ be a convergent approximation.*

4 Hyperbolic Systems of Conservation Laws

The fundamental nature of hyperbolic conservation laws can easily be seen from the following formal derivation. Consider a conserved quantity with density $u = u(x, t)$, where x denotes the space variable and t denotes time. Assume that the quantity moves with velocity $v = v(x, t)$. Conservation yields that

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = -\int_{\partial\Omega} u(x,t)v(x,t)\cdot n(x,t)\,dS, \tag{4.1}$$

where $\Omega \subset \mathbb{R}^d$ is some fixed domain in space with boundary $\partial\Omega$ and outward unit normal $n(x, t)$. Gauss' theorem implies that

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = -\int_\Omega \nabla_x \cdot \big(u(x,t)v(x,t)\big)\,dx, \tag{4.2}$$

which rewrites to

$$\int_\Omega \Big(u(x,t)_t + \nabla_x \cdot \big(u(x,t)v(x,t)\big)\Big)\,dx = 0, \tag{4.3}$$

from which we conclude that

$$u(x,t)_t + \nabla_x \cdot \big(u(x,t)v(x,t)\big) = 0. \tag{4.4}$$

Making the assumption that the velocity v depends on the density solely, we introduce the flux function $f(u) = uv(u)$ to find the hyperbolic conservation law

$$u_t + \nabla_x \cdot f(u) = 0. \tag{4.5}$$

There was nothing in the previous derivation that prevented the quantity u from being a vector, $u \in \mathbb{R}^n$, in which case we have a system of hyperbolic conservation laws. A particular case of these equations is the Euler equations of gas dynamics. In spite of the fundamental nature of these equations, there is no general theory unless

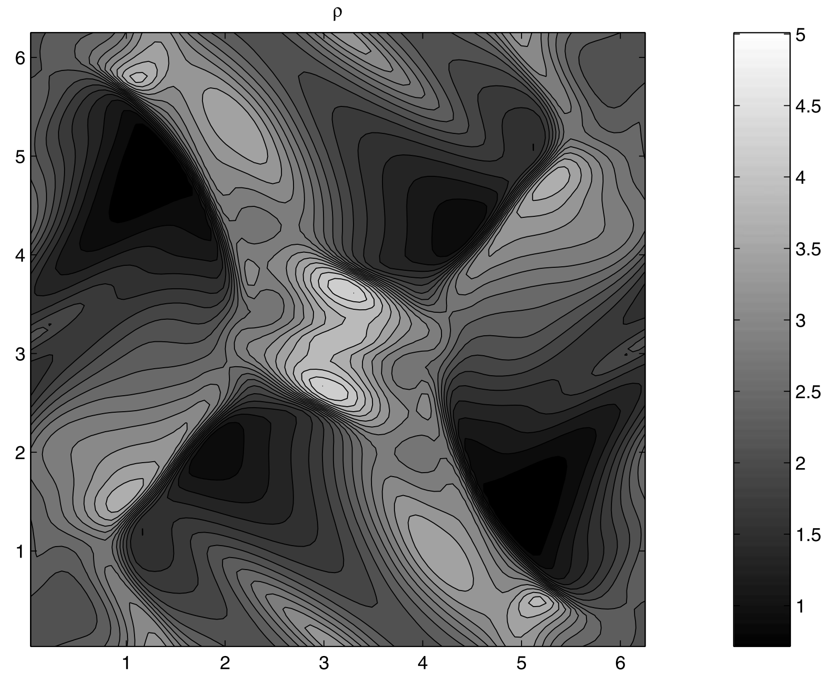


Fig. 1 The figure shows the density in a computation in magnetohydrodynamics (MHD) using the *HLL method* [L126]. MHD can be described by a 9×9 system of hyperbolic conservation laws. The solution of the Riemann problem is not completely known, and one has to resort to numerical simulations

the spatial dimension equals one, $d = 1$, or the conservation law is scalar, $n = 1$. In Lax's words [L224] "There is no theory for the initial value problem for compressible flows in two space dimensions once shocks show up, much less in three space dimensions. This is a scientific scandal and a challenge". Numerical computations of solutions of these equations are notoriously difficult due to the intrinsic occurrence of discontinuities, denoted shocks, in the solution, even for smooth initial data. See, e.g., Fig. 1. But [L224] "Just because we cannot prove that compressible flows with prescribed initial values exist doesn't mean that we cannot compute them". Already Riemann observed that the discontinuous solutions were physical, and could not be dispensed with. The state of affairs in this area at the end of World War II, when Peter Lax entered the scene, is nicely summarized in the book [10].

Thus one has to use the notion of weak or distributional solutions, and that suddenly makes uniqueness of solutions a difficult issue. We say that $u \in L^1(\mathbb{R} \times [0,\infty))$ is a weak solution of (4.5) with given initial data $u|_{t=0} = u_0 \in L^1(\mathbb{R})$ if

$$\int_0^\infty \int_{\mathbb{R}} \big(u\phi_t + f(u) \cdot \nabla_x \phi\big)\, dx\, dt + \int_{\mathbb{R}} u_0 \phi|_{t=0}\, dx = 0 \tag{4.6}$$

for all compactly supported and smooth functions ϕ. Criteria to identify the unique physical solution among the multitude of possible weak solutions are denoted entropy conditions.

One approach to resolve the uniqueness issue has been to consider the equation (4.5) as an approximation of a model that includes diffusion. Thus one accepts as solutions of (4.5) only those functions u that are limits of solutions u^ϵ, i.e., $u = \lim_{\epsilon \downarrow 0} u^\epsilon$, of the viscous regularization

$$u_t^\epsilon + \nabla_x \cdot f(u^\epsilon) = \epsilon \Delta_x u^\epsilon. \tag{4.7}$$

Let us first discuss the Cauchy problem in the scalar case on the line, i.e., $n = d = 1$, which we write as

$$u_t + f(u)_x = 0, \quad u|_{t=0} = u_0. \tag{4.8}$$

In a key paper from 1950, Hopf [36] analyzed the viscous Burgers equation, $u_t + uu_x = \epsilon u_{xx}$, i.e., the scalar one-dimensional equation (4.7) with $f(u) = u^2/2$, in the inviscid limit, that is, as $\epsilon \to 0$, and described the limit. Using the Cole–Hopf transform, the equation could be rewritten as the heat equation. The paper by Hopf represents the start of the mathematical theory of conservation laws, and the theory developed rapidly in several mathematical directions in the US, the Soviet Union, and China.

Lax introduced in [L10] what has subsequently been called the *Lax–Friedrichs difference scheme*,[5] which can be defined as follows. Let Δx, Δt be (small) positive numbers, and denote the approximate solution to u at $(j\Delta x, n\Delta t)$ by u_j^n, i.e., $u_j^n \approx u(j\Delta x, n\Delta t)$. Given a discretization of the initial data, i.e., given u_j^0 for $j \in \mathbb{Z}$, one can use the recursive definition[6]

$$u_j^{n+1} = \frac{1}{2}\big(u_{j-1}^n + u_{j+1}^n\big) - \frac{\Delta t}{2\Delta x}\big(f(u_{j+1}^n) - f(u_{j-1}^n)\big) \tag{4.9}$$

to compute u_j^n for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $j \in \mathbb{Z}$. The scheme (4.9) constitutes the Lax–Friedrichs scheme,[7] and it is stable if the Courant–Friedrichs–Lewy (CFL) condition (see also [L55], [L204]), $\|f'\|_\infty \Delta t/\Delta x < 1$, holds. Being so simple to implement, and having been so well-studied over a period of more than half a century, it is still one of the first methods to be employed for simple numerical tests.

---

[5] Friedrichs had studied the same scheme for linear symmetric hyperbolic equations.

[6] A straightforward Taylor expansion for smooth solutions shows that the difference scheme is consistent with the differential equation. Clearly it is necessary for a difference scheme to be consistent in order to be useable. However, it is far from being sufficient for it to converge to the right solution.

[7] Peter writes [p. 227, L203]: "Numerical experiments is the life-blood of the investigation of numerical methods. My first numerical studies of the Lax–Friedrichs scheme were done in 1954, before the advent of compilers, using von Neumann's MANIAC at the Los Alamos Laboratory, built under the tutelage of Nick Metropolis. It had a huge memory of 1,024 words and used punched cards, like the voters in Palm Beach County".

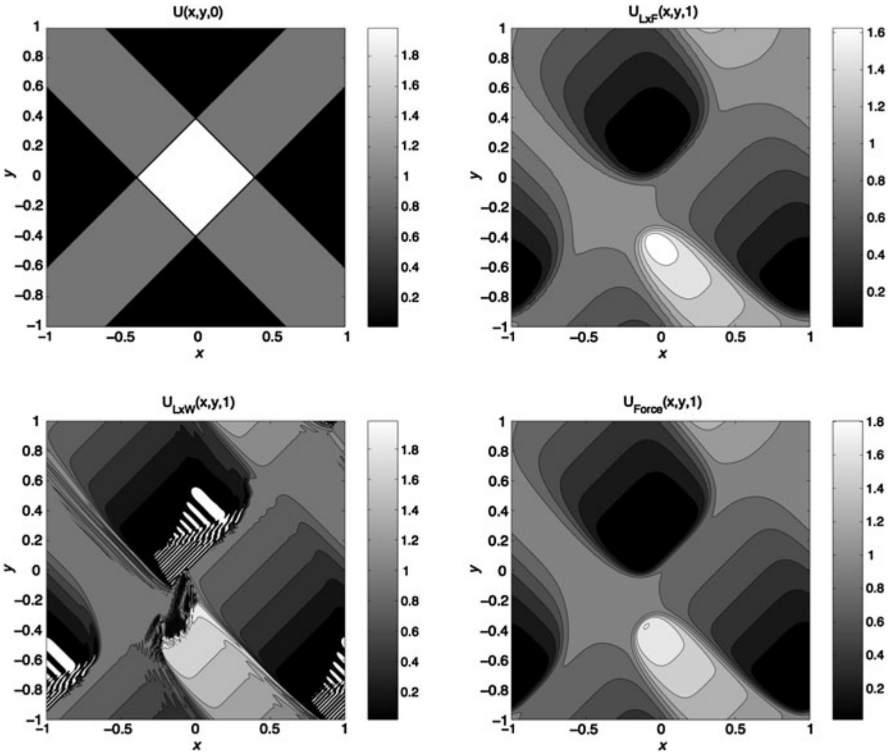


Fig. 2 The figures show the solution of the equation $u_t + \frac{1}{2}(u^2)_x + (\sin u)_y = 0$ with initial condition as in the upper left-hand corner. In the upper right-hand corner the solution is depicted using the Lax–Friedrichs scheme. The lower left-hand corner shows the solution using the Lax–Wendroff scheme, and the oscillations are clearly visible. In the lower right-hand corner the solution using a combination of the Lax–Friedrichs and the Lax–Wendroff scheme is depicted

See, e.g., Fig. 2. Of course, the theory of numerical methods for these equations has developed dramatically over this period, and a survey of current methods can be found in [46]. The convergence of the Lax–Friedrichs scheme was proved by Oleĭnik in [55] (see also [32] and [67]), in which she also introduced the Oleĭnik entropy condition [56], which is fundamental in all further study of asymptotic behavior of solutions of scalar conservation laws. Furthermore, she proved the convergence of the vanishing viscosity method for scalar conservation laws in one space dimension. Extensions to several space dimensions have been given by Conway and Smoller [9]. The proof of convergence of the Lax–Friedrichs scheme for isentropic gas dynamics was accomplished in [6, 16–18].

The paper [L20] constitutes a cornerstone in the theory of systems of hyperbolic conservation laws. Here Lax undertakes the first comprehensive mathematical study of the case of systems. The first part addresses the scalar case, and we will start by discussing that part. Here Lax derives what has later been denoted the *Lax formula*: Assume that f is strictly convex and u_0 is bounded. Then the weak entropy solution of (4.8) is given by $u = v_x$ where

$$v(x,t) = \sup_{y\in\mathbb{R}}\left(v_0(y) + tf^*\left(\frac{x-y}{t}\right)\right). \tag{4.10}$$

Here v_0 is a primitive of u_0 and f^* is the convex conjugate function of f (see [66, Thm. 2.5.1]). In [Sect. 5, L20] Lax studies the asymptotic behavior as $t \to \infty$ of solutions of scalar conservation, both on the full line and the periodic case. This paper was the starting point for the extensive generalization by Glimm and Lax in their deep analysis, [L52] and [L66], of the asymptotic behavior of solutions of 2×2 systems of conservation laws. There has always been a strong connection between the theory for hyperbolic conservation laws and Hamilton–Jacobi equations. The Lax formula played a decisive role in the work of P.L. Lions, Crandall, and Kružkov on the Hamilton–Jacobi equations.

Next, consider the case of one-dimensional systems, i.e., $n > 1$ and $d = 1$. The seminal paper [L20] sets the stage for all subsequent development in this area; the main concepts are introduced, and the paper culminates with the *Lax theorem*, i.e., the solution of the Riemann problem for systems of hyperbolic conservation laws on the line. Before we can state the theorem, we have to define the basic quantities. We consider the equation (4.5), where now u and f are vectors in $\mathbb{R}^n$. The Jacobian $df(u)$ is an $n \times n$ matrix assumed to have, in the strictly hyperbolic case, n real and distinct eigenvalues $\lambda_j(u)$ (assumed ordered) with corresponding eigenvectors $r_j(u)$, viz.

$$df(u)r_j(u) = \lambda_j(u)r_j(u), \quad j = 1, \dots, n. \tag{4.11}$$

The jth wave family is said to be linearly degenerate if $\nabla\lambda_j(u) \cdot r_j(u) = 0$ and genuinely nonlinear if $\nabla\lambda_j(u) \cdot r_j(u) \neq 0$.[8] Weak solutions are defined as in the scalar case, however the question of the appropriate entropy condition is considerably more complicated for systems. Riemann analyzed in the fundamental paper [64] the following physical problem: Consider a tube with two gases separated initially by a membrane. Removing the membrane yields a Cauchy problem with initial data

$$u|_{t=0} = u_0 = \begin{cases} u_l & \text{for} \quad x < 0, \\ u_r & \text{for } x \geq 0. \end{cases} \tag{4.12}$$

Here u_l and u_r are constants, and the unknown vector u denotes the densities of the conserved quantities. The Cauchy problem for (4.8) with initial data (4.12) is called the Riemann problem. In his study of the problem, Riemann also conserved the entropy, which, while formally derivable from the conservation laws for smooth solutions, does not hold for weak solutions, see [67, p. 264] and [L165]. It was in [L20] that a complete solution was first given in the case of small initial data. There

[8] Both cases occur for the Euler equations. Systems where a wave family changes from linearly degenerate to genuinely nonlinear, are not discussed here.

are several classes of solutions, and the full solution of the Riemann problem is composed of so-called simple waves. A shock solution has the form

$$u(x,t) = \begin{cases} u_l & \text{for } x < st, \\ u_r & \text{for } x \geq st, \end{cases} \tag{4.13}$$

where the velocity has to satisfy the Rankine–Hugoniot relation

$$s(u_r - u_l) = f(u_r) - f(u_l), \tag{4.14}$$

in order to be a weak solution.[9] However, shocks are abundant, and additional requirements are needed to eliminate unphysical solutions. Lax gave the correct entropy condition, now denoted the *Lax entropy condition*, which states that (4.13) is a j entropy shock if the inequalities

$$\lambda_{j-1}(u_l) < s < \lambda_j(u_l), \qquad \lambda_j(u_r) < s < \lambda_{j+1}(u_r) \tag{4.15}$$

hold. Such shocks are denoted *Lax shocks*. Furthermore, in addition to shock solutions, the Riemann problem (4.12) may, in the genuinely nonlinear case, have continuous solutions called rarefaction waves. These are solutions of the form

$$u(x,t) = \begin{cases} u_l & \text{for } x \leq \lambda_j(u_l)t, \\ w(x/t) & \text{for } \lambda_j(u_l)t < x < \lambda_j(u_r)t, \\ u_r & \text{for } x \geq \lambda_j(u_r)t, \end{cases} \tag{4.16}$$

where the function $w = w(\xi)$ satisfies

$$\dot{w}(\xi) = r_j(w(\xi)), \qquad \nabla\lambda_j w(\xi) \cdot \dot{w}(\xi) = 1. \tag{4.17}$$

In the case of a linearly degenerate wave family, we only have discontinuous solutions, denoted contact discontinuities, that equal[10]

$$u(x,t) = \begin{cases} u_l & \text{for } x < st, \\ u_r & \text{for } x \geq st, \end{cases} \tag{4.18}$$

where the speed s satisfies the Rankine–Hugoniot condition, and we have $s = \lambda_j(u_l) = \lambda_j(u_r)$. Shocks, rarefaction waves, and contact discontinuities constitute the simple waves, and the solution of the Riemann problem is a concatenation of simple waves. We can now state the Lax theorem.

Theorem 4.1 *Assume that the system (4.8) is strictly hyperbolic and that each wave family is either genuinely nonlinear or linearly degenerate. Consider a* $u_l \in \mathbb{R}^n$.

[9]This relation simply states that conservation holds across discontinuities.

[10]Contact discontinuities look like shocks, but they do not satisfy the Lax entropy condition.

Then there exists a neighborhood Ω of u_l such that for each $u_r \in \Omega$ the Cauchy problem with initial data (4.12) *has a unique weak solution consisting of up to $n+1$ constant states separated by shocks*, *rarefaction waves and contact discontinuities*.

This theorem is still as central in the theory of conservation laws now as it was half a century ago. Furthermore, Lax provides a comprehensive study of the notion of Riemann invariants in [L20].

In the pioneering paper [L75], published in a conference proceedings, Lax takes a novel and penetrating look at entropies. He says that a convex function $U(u)$ is an entropy for the system (4.8) if all smooth solutions satisfy another conservation law of the form

$$U(u)_t + F(u)_x = 0. \tag{4.19}$$

A necessary and sufficient condition for this is that

$$U_u f_u = F_u, \tag{4.20}$$

which is an underdetermined system for $n = 1$, determined for $n = 2$, and overdetermined for $n > 2$ (however, for compressible flow it does have a solution). Let us introduce a dissipative regularization

$$u^\epsilon_t + f(u^\epsilon)_x = \epsilon u^\epsilon_{xx}. \tag{4.21}$$

Multiply this with U_u and rearrange to find

$$U(u^\epsilon)_t + F(u^\epsilon)_x = \epsilon(U(u^\epsilon)_{xx} - u^\epsilon_x U(u^\epsilon)_{uu} u^\epsilon_x) \le \epsilon U(u^\epsilon)_{xx}, \tag{4.22}$$

using that U was assumed to be convex. Let $\epsilon \to 0$, assuming $u^\epsilon \to u$ strongly, which yields

$$U(u)_t + F(u)_x \le 0 \tag{4.23}$$

weakly. We say that u is a weak entropy solution if (4.23) is satisfied for all convex entropies U. In the scalar case in several space dimensions, Kružkov [45] had shown that it sufficed to consider the family of entropies $U(u) = |u - k|$ for all $k \in \mathbb{R}$, in which case $F(u) = \operatorname{sgn}(u - k)(f(u) - f(k))$. Observe that (4.23) introduces a direction of time, and makes the solution time irreversible, which is in contrast to the situation for linear systems. Next, Lax applies the same approach to difference schemes in a way that turned out to be instrumental in the further development of numerical schemes by, e.g., Wendroff, Harten, Hyman, Osher, Majda, Engquist, and Tadmor.

In [L29], Lax establishes with Wendroff what is now known as the *Lax–Wendroff theorem*: Consider an approximate solution v to the solution u of (4.8) computed by a conservative and consistent finite difference scheme with discretization parameters Δt and Δx. Assume that as $\Delta t, \Delta x \to 0$ the approximation v converges boundedly almost everywhere to some function u. Then u is a weak solution of (4.8).

In [L29] and a subsequent paper [L43], Lax and Wendroff analyze what is now known as the *Lax–Wendroff difference scheme* (see Fig. 2), which is a second-order scheme. Convergence of the scheme is proved in [7].

There has of course been an extensive development in the mathematical theory for systems of hyperbolic conservation laws since Lax's landmark solution of the Riemann problem. Most prominent is Glimm's solution of the general Cauchy problem by the random choice method in the case of small variation in the initial data [34] (see also [L116] and [L198]). Indeed to solve the Cauchy problem, one makes a piecewise constant approximation of the initial data, turning it into a multiple Riemann problem. In order to be able to apply Lax's theorem, each jump has to be small, implying the restriction of small total variation. The restriction of small total variation is, except for special systems, a deep part of the theory, and intrinsically linked with hyperbolic conservation laws. The question of uniqueness of the weak entropy solution and the existence of a continuous semigroup of a system of hyperbolic conservation laws on the line remained open for a long time, and it was only fairly recently resolved by Bressan et al. [4] (see also [11] and [35] for a survey) using the method of (wave) front tracking. Finally, the proof of convergence of the viscous limit for systems of hyperbolic conservation laws was obtained by Bianchini and Bressan [3]. Thus one can say that with these results, the theory for solutions with small total variation of systems of hyperbolic conservation laws on the line has become mature. A comprehensive survey of the current state of affairs can be found in [11].

Speed depends on size
Balanced by dispersion
Oh solitary splendor
PETER D. LAX

5 Integrable Systems

Most surveys on integrable systems start with the story of the Scottish engineer John Scott Russell and his discovery of solitary waves in the canals outside Edinburgh in 1834. We refer to [5] for the historical background, and we enter the history at a later stage. In 1895, Korteweg and de Vries derived a nonlinear partial differential equation, now universally known as the KdV equation,[11] that models surface waves in shallow water, and that described the phenomena observed by Scott Russell. Furthermore, they showed that the equation had soliton solutions, a term coined by Zabusky and Kruskal [73] in 1965.

The landmark paper in the theory of integrable systems is the Gardner, Greene, Kruskal, and Miura paper [30] from 1967 where they solve the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad u|_{t=0} = u_0 \tag{5.1}$$

[11]The KdV equation had already been derived in 1871 by Boussinesq, see [61].

by the inverse scattering transform (IST). We can describe the IST as follows. Given initial data such that $\int_{\mathbb{R}}(1+x^2)|u_0(x)|\,dx < \infty$, assume that $u = u(x,t)$ is a solution of the KdV equation. We consider the second order ordinary differential operator[12] $L(t) = -D^2 + u(x,t)$ (writing $D = \frac{d}{dx}$) where t acts as a parameter, and where $u(x,t)$, called the potential, is the solution of the KdV equation. For $L(t)$ one computes the forward (direct) scattering data, that is, the reflection coefficient $R(k,t)$, transmission coefficient $T(k,t)$, the (negative) eigenvalues $\lambda_1(t), \dots, \lambda_n(t)$, and the (positive) norming constants $c_1(t), \dots, c_n(t)$ of appropriately chosen eigenvectors that are square integrable. When the potential $u(x,t)$ satisfies the KdV equation, these quantities have an amazingly simple and explicit behavior in the t-variable:

$$\begin{aligned} &R(k,t) = R(k,0)e^{8ik^3t}, && T(k,t) = T(k,0), \\ &\lambda_j(t) = \lambda_j(0), && c_j(t) = c_j(0)e^{4(-\lambda_j)^{3/2}t}, \quad j = 1, \dots, n. \end{aligned} \tag{5.2}$$

Thus the IST works as follows: Compute the forward scattering data for the initial data, let the scattering data evolve in t according to (5.2), and use inverse scattering to determine the potential $u(x,t)$, which then is the required solution of the KdV equation. For the inverse scattering one has the very powerful machinery of Marchenko. It was known that the KdV equation possessed multi-soliton solutions, i.e., localized solutions that interacted almost particle-like, and that the KdV equation had infinitely many conserved quantities. See, e.g., [29, 31].

A big question mark with the IST was the relationship between the KdV equation and the linear ordinary differential equation. Shortly after the IST appeared in [30], Lax published [L60] where he introduced what ever after has been called the *Lax pairs* that unveiled the underlying mechanism for the series of coincidences that appeared for the KdV equation. The key derivation is so short that we can reproduce it here. Consider a one-parameter family $u(t)$ of functions in a space $\mathcal{B}$, and suppose that to each $u \in \mathcal{B}$ we can associate a self-adjoint operator $L(u)$ such that if u satisfies the evolution equation

$$u_t = K(u), \tag{5.3}$$

the operator $L(u(t))$, which we for brevity denote $L(t) = L(u(t))$, remains unitarily equivalent. The unitary equivalence implies the existence of a unitary operator $U(t)$ such that

$$U(t)^{-1}L(t)U(t) \tag{5.4}$$

is time independent. Taking the time derivative in (5.4), we infer

$$-U^{-1}U_tU^{-1}LU + U^{-1}L_tU + U^{-1}LU_t = 0. \tag{5.5}$$

[12] We recognize this operator as the one particle Schrödinger Hamiltonian in nonrelativistic quantum mechanics, but this aspect plays little role in the following except for the fact that the operator has been extensively studied.

A one-parameter family $U(t)$ of unitary operators satisfies

$$U_t = PU \tag{5.6}$$

for some anti-symmetric operator $P = P(t)$. Rewriting (5.5) we find the *Lax relation*

$$L_t = [P, L] \tag{5.7}$$

where $[\cdot, \cdot]$ is the commutator and L, P is the *Lax pair*.[13] How does this relate to the KdV equation? First of all, we have $K(u) = 6uu_x - u_{xxx}$. In [30] Gardner, Greene, Kruskal, and Miura had discovered that if u satisfies the KdV equation, then the eigenvalues of the linear operator $L = -D^2 + u$ remain invariant with respect to time. Defining the operator

$$P = -4D^3 + 3uD + 3Du, \tag{5.8}$$

a simple computation reveals that

$$L_t - [P, L] = u_t - 6uu_x + u_{xxx}, \tag{5.9}$$

which is nothing but the KdV equation, rather than a complicated fifth order differential operator. Considerable guesswork was involved here: Given a nonlinear partial differential equation (the KdV equation), could one find a linear operator L and an antisymmetric operator P such that the Lax relation (5.7) is equivalent to the equation itself? In 1972, Zakharov and Shabat showed [75] that also the nonlinear Schrödinger equation possessed a Lax pair, and with a different associated linear operator. This was important because it showed that the Lax pair was not an artifact of the KdV equation, but could eventually be applied to other nonlinear partial differential equations as well. Shortly thereafter, they introduced the method of zero-curvature equations [76, 77], as another means to establish complete integrability. In the zero-curvature formalism one considers a vector-valued function $\Psi = \Psi(x, t, \lambda)$ (where λ is a spectral parameter), denoted the Baker–Akhiezer function, that simultaneously satisfies two ordinary differential equations

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi \tag{5.10}$$

for given matrices $U = U(u, \lambda)$ and $V = V(u, \lambda)$. The equality of mixed derivatives, i.e., $\Psi_{xt} = \Psi_{tx}$, implies that

$$U_t - V_x + [U, V] = 0, \tag{5.11}$$

which is the zero-curvature relation. The aim is to construct matrices U and V in such a way that (5.11) reduces to a given partial differential equation. In the case of

[13] We follow Gel'fand and write P (Peter) and L (Lax) for the Lax pair. In [L60] Lax used B to denote the operator P.

the KdV equation, one choice is

$$U = \begin{pmatrix} 0 & 1 \\ -\frac{1}{4}\lambda + u & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} -u_x & \lambda + 2u \\ (-\frac{1}{4}\lambda + u)(\lambda + 2u) - u_{xx} & u_x \end{pmatrix} \tag{5.12}$$

such that

$$U_t - V_x + [U, V] = \begin{pmatrix} 0 & 0 \\ u_t - 6uu_x + u_{xxx} & 0 \end{pmatrix} = 0, \tag{5.13}$$

which is equivalent with the KdV equation. The discovery of the Lax pair and the zero-curvature formalism started an immense race to study the key equations of mathematical physics to decide if they are completely integrable in the sense of existence of Lax pairs or zero-curvature formalism. By now many of the main equations have been determined to be completely integrable, e.g., the nonlinear Schrödinger equation, the sine-Gordon equation, the AKNS system (developed by Ablowitz, Kaup, Newell, and Segur), the modified KdV equation, the Kadomtsev–Petviashvili equation, the Boussinesq equation, the Thirring equation, the Camassa–Holm equation, the Toda lattice, the Kac–van Moerbeke lattice, and the Ablowitz–Ladik system, and an inverse scattering transform formalism has been set up for each of them.

In the very same paper [L60], Lax observes that the operator B is not the unique antisymmetric operator with the property that $L_t - [B, L]$ is a pure multiplication operator (as opposed to a differential operator); by carefully constructing odd order differential operators B_{2n+1}, Lax could construct a hierarchy (now called the *Lax hierarchy* for the KdV equation) of nonlinear partial differential equations with similar properties to the KdV equation, for instance, the whole hierarchy possesses a Hamiltonian structure, as proved by Zakharov and Faddeev [74] for the KdV equation. The Lax hierarchy was the starting point in establishing the very rich connections between nonlinear partial differential equations and algebraic geometry. See, e.g., [2, 33, 54] for a survey. It is difficult to overestimate the importance the notion of Lax pairs;[14] hardly a paper can be written on integrable systems without describing its Lax pair or its zero-curvature formulation.[15]

In papers [L89] and [L91], Lax studies periodic solutions of the KdV equation. He constructs a large family of special solutions, periodic in x and almost periodic in t, and study their behavior. His methods were direct, using only calculus in function spaces, properties of the Lax hierarchy, and the existence of infinitely many conserved quantities, thereby revealing many of the fundamental properties of completely integrable systems. This was in a period of very rapid development, and where the powerful machinery of algebraic geometry was applied to these equations. We refer to papers by Dubrovin and Novikov [19–22, 52, 53], Its and Matveev [41, 42], and McKean and van Moerbeke [50].

[14] In Lax's own words [p. 234, L203] "[Lax pairs] occur in surprisingly many contexts".

[15] An unscientific and informal indication of the importance is given if you google on "Lax pair", you come up with more than 25,000 hits (Jan 25, 2008).

The *Lax–Levermore theory* ([L128], [L129], and [L130], and announced in [L112]) concerns the vanishing dispersion limit of the KdV equation, and its represents a landmark in our understanding of dispersive waves and the inverse scattering transform. In the words of P. Deift [p. 611, L215] "The papers by Lax and Levermore constitute one of the earliest successes in turning the inverse scattering transform into a tool for detailed asymptotic analysis". We have seen above that if one considers the inviscid limit of the viscous Burgers equation $u^\epsilon_t + u^\epsilon u^\epsilon_x = \epsilon u^\epsilon_{xx}$, that is, consider the limit $u = \lim_{\epsilon\to 0} u^\epsilon$, one gets the correct entropy solution of $u_t + uu_x = 0$. The Lax–Levermore theory deals with the same question when one replaces the viscous regularization of the inviscid Burgers equation by the dispersive regularization (which then is the KdV equation). Thus we are interested in the weak limit $\epsilon \to 0$ of solutions of

$$u^\epsilon_t - 6u^\epsilon u^\epsilon_x + \epsilon u^\epsilon_{xxx} = 0 \tag{5.14}$$

with initial data $u|_{t=0} = u_0$.

It is rather straightforward to see that the weak limit of the vanishing dispersion equation cannot equal the weak limit of the vanishing diffusion equation. We follow the argument of [L155]: Denote by $\bar{u}$ the weak limit of u^ϵ, the solution of (5.14), as $\epsilon \to 0$. If we rewrite (5.14) in conservative form, it reads

$$u^\epsilon_t - 3\big((u^\epsilon)^2\big)_x + \epsilon u^\epsilon_{xxx} = 0. \tag{5.15}$$

The first and the last term have weak limits equal to $\bar{u}_t$ and zero, respectively, and hence the middle term has a limit as well, which we write as

$$\overline{u^2} = \text{w-lim}_{\epsilon\to 0}(u^\epsilon)^2, \tag{5.16}$$

implying that

$$\bar{u}_t - 3\overline{u^2}_x = 0. \tag{5.17}$$

It is known that if the limit is weak, but not strong we have

$$\overline{u^2} > \bar{u}^2. \tag{5.18}$$

Inserting this into (5.17) we see that

$$\bar{u}_t - 3((\bar{u})^2)_x \neq 0, \tag{5.19}$$

unless $\overline{u^2}$ differs by $\bar{u}^2$ by a constant, which can be ruled out by a separate argument.

The question of the behavior of the vanishing dispersion limit turned out to have a highly nontrivial and very interesting answer using the inverse scattering transform for the KdV equation.

Recall that the scalar conservation law $u_t - 6uu_x = 0$ develops singularities, or shocks, in finite time for generic smooth initial data. The first result says that for times before singularities occur, we have that u^ϵ converges to u, the solution of $u_t - 6uu_x = 0$. However, for times after the classical solution ceases to exist, new phenomena occur. Lax and Levermore show that u^ϵ converges weakly to $\bar{u}$, i.e., $u^\epsilon \rightharpoonup \bar{u}$, in $L^2(\mathbb{R})$, where

$$\bar{u} = \partial_{xx} Q^* \tag{5.20}$$

with $Q^* = Q^*(x,t)$ determined by

$$Q^*(x,t) = \min_{0\le\psi\le\phi} Q(\psi; x, t). \tag{5.21}$$

The function ϕ equals

$$\phi(\eta) = \mathrm{Re}\int \frac{\eta}{(-u(y)-\eta^2)^{1/2}}\,dy \tag{5.22}$$

and $Q(\psi; x, t)$ is a complicated, explicitly given, function that is linear in x and t while quadratic in ψ. The function $u_0 \in C^1(\mathbb{R})$ is assumed to be nonpositive, and have finitely many critical points. Venakides [69] has extended the theory to include positive initial data. The periodic case is treated in [70] and [71]. The fact that u_0 is nonpositive is used to replace it by another function u_0^ϵ with vanishing reflection coefficient and such that $u_0^\epsilon \to u_0$ in $L^2(\mathbb{R})$ as $\epsilon \to 0$. A nice survey of the theory up to 1993 can be found in [L171].

A major step forward was taken when Deift and Zhao [14] developed a novel steepest-decent method for Riemann–Hilbert problems with oscillatory coefficients. Shabat had observed that the inverse scattering transform could be viewed as a Riemann–Hilbert problem. This paved the way for the important result by Deift, Venakides, and Zhao [13] where they applied the results of [14] to determine all coefficients in the asymptotic expansion.

The Lax–Levermore–Venakides theory has also been employed in a number of problems involving asymptotic analysis, we here mention the following: the semi-classical limit of the defocusing nonlinear Schrödinger equation [43], the Toda shock problem [72], the continuum limit of the Toda lattice [12].

6 Lax–Phillips Scattering Theory

The collaboration of Lax and Phillips started in 1960 and lasted for over 30 years. It rivals any of the great mathematical collaborations of the 20th century. It had been suggested to both of them on different occasions that theirs was a lot like the Hardy–Littlewood collaboration and in both cases the immediate response was which of them was Hardy and which Littlewood? The primary focus of their joint works is the study of solutions to the wave equation in geometric settings, specifically unbounded domains in Euclidean spaces and on hyperbolic manifolds of finite and infinite volume. Their analysis introduced a mixture of tools from linear hyperbolic partial differential equations, functional analysis and the lovely interplay of these with the geometry of the domain. This continues today to be a central topic in geometric analysis.

The starting point of their collaboration is the paper [L32] establishing the decay of energy for solutions to the wave equation in the exterior of a smooth bounded domain O in $\mathbb{R}^3$. Moreover they show that asymptotically the solutions behave like solutions to the wave equation in free space. This was followed by the joint paper [L33] with Morawetz in which they show in the exterior of a convex domain, that locally the energy decays exponentially as t goes to infinity. These papers contain the elements of the abstract, time dependent, scattering theory developed by Lax and Phillips. This theory is similar in spirit to the abstract spectral theorem but it takes into account via its "incoming and outgoing" subspaces and corresponding completeness statements, the geometry at infinity which is critical to the understanding of the finer features of solutions to the wave equation. A comprehensive treatment of their Euclidean Scattering Theory is given in their well known monograph [L53]. The revised edition of [L53] from 1989 contains an up-to-date epilogue with a wealth of information about recent developments stemming from their earlier works. In their paper [L42] one finds far-reaching insights and Conjectures about the relation between the motion of rays in the exterior of the obstacle O and the distribution of the scattering poles. The latter are identified in terms of the eigenvalues of the fundamental compact (not self-adjoint) operator B which is the infinitesimal generator of the semi-group $Z(t)$ associated with the abstract setup of the scattering problem. O is said to be non-trapping if there is a ball C in $\mathbb{R}^n$ containing O and a time t_0 such that any ray entering C and obeying linear motion outside of O and standard reflection on hitting O, exits C within the time t_0. A basic conjecture put forth in [L42] is that the semi-group $Z(t)$ is eventually compact (which is equivalent to the imaginary parts of the poles of the scattering matrix tending to infinity) if and only if O is non-trapping. If O is trapping this conjecture was established by Ralston [63]. In the other direction Ludwig and Morawetz [47] established the Conjecture if O is convex. After progress on some other special cases the general case of the Conjecture was proven by Melrose in [48]. His work makes use of the modern machinery of propagation of singularities for hyperbolic equations and related microlocal analysis. Not coincidentally this modern theory [39] has its roots in Lax's 1957 paper [L23] mentioned in Sect. 2.

A further study of the distribution of the poles of the scattering operator is contained in [L61], in analogy with the well studied Weyl Law for the counting of eigenvalues of the Laplacian on a compact domain. However unlike that case the poles here are not confined to lie on a line and this complicates the problem considerably. Lax and Phillips determine the order of magnitude of the number of purely imaginary poles. More recently Melrose and Zworski [49, 78] have estimated the number of poles in a large ball while Ikawa [40] gives a precise description of the location of the poles near the real axis in the case that O consists of two convex bodies containing an unstable ray between them. All of these topics remain active areas of investigation even today.

Faddeev and Pavlov [27] observed that Lax and Phillips's abstract set up for scattering theory in Euclidean spaces is also well suited in the context of non-compact but finite area hyperbolic surfaces. This led Lax and Phillips to develop their theory in this context and the outcome was a series of far-reaching papers on this and related topics. The corresponding spectral theory for finite volume hyperbolic manifolds is due to Selberg [65]. His main results being the analytic continuation of Eisenstein series and the development of the trace formula. In [L77] and [L91], Lax and Phillips develop their scattering theory approach for the continuous spectrum on these manifolds. Their method is based on their semi-group $Z(t)$ and its infinitesimal generator B. It yields a short and conceptual proof of the analytic continuation of the Eisenstein series. It also identifies the poles of the latter in terms of the eigenvalues of the compact operator B. This is particularly well suited for studying the behaviour of these poles when the surface is deformed. The Lax and Phillips theory was used in [62] as a basis for such a study. The cut-off Laplacian (associated with a cusp of the manifold) was introduced in the monograph [L77] and in the hands of Colin de Verdiére [8] and Müller [51], it is a fundamental tool in the proof of the trace class conjecture for the spectrum of a general locally symmetric space.

Papers [L103] and [L115] and the series [L116], [L127], [L128], [L134] are concerned with the Radon transform in non-Euclidean spaces, and related Paley–Wiener theorems. These are developed in order to carry out their scattering theory for infinite volume but geometrically finite hyperbolic manifolds. Prior to these papers little was known about these. For the case of surfaces Patterson [57–59] developed the spectral theory but his methods were limited to two dimensions. Lax and Phillips establish the basic properties of the spectrum, that is, the finiteness of the point spectrum below the continuous spectrum and the absolute continuity of the rest of the continuous spectrum. Their series of papers on translation representations for solutions of the wave equation for these manifolds yield a complete spectral and scattering theory. They stop short of obtaining a trace formula in this context. One may view the more recent work [60] as a substitute for the trace formula in this infinite volume context.

The paper [L116] investigates the analogue of the well known problem of Gauss of counting asymptotically the number of lattice points in a large circle, in the context of Euclidean and hyperbolic spaces. It was shown by Delsarte [15] some time ago that for the case of the hyperbolic plane with a co-compact lattice acting isometrically, the spectrum of the Laplacian on the quotient can be used to obtain the leading term of the count of the number of lattice points in a large hyperbolic disk. In [L116], Lax and Phillips use the wave equation and their theory to obtain the asymptotics of lattice points in a large ball, for any geometrically finite group. The remainder term that they obtain for this count is the sharpest such result known in all cases. Before their work, Selberg in unpublished work had obtained similar results in the finite volume case. These counting problems can be generalized to other symmetric spaces and have Diophantine applications. This is an active area of current research (see, for example, [25] and [28]).

More recently in papers [L195], [L199] and [L200] with Francsics, Lax gives an explicit fundamental domain for the Picard group in $\mathrm{SU}(2,1)$. This opens the way for a numerical as well as a concrete investigation of the spectral theory of such quotients.

To end this brief survey of Lax's work in scattering theory and functional analysis we mention his 2002 text *Functional Analysis*, [L187]. There are many good texts on this subject, however this one with its many examples, enticing applications and clear development of the theory, is a classical. It gives students and researchers an excellent opportunity to learn this basic subject from one of its masters.

Acknowledgements HH is grateful to Nils Henrik Risebro for providing the figures, and to Constantine Dafermos, Fritz Gesztesy, Harald Hanche-Olsen, and Kenneth H. Karlsen for careful reading of the manuscript

References

1. Albers, D.J., Alexanderson, G.L., Reid, C. (eds.): More Mathematical People. Hartcourt Brace Jovanovich, Boston (1990)
2. Belokolos, E.D., Bobenko, A.I., Enol'skii, V.Z., Its, A.R., Matveev, V.B.: Algebro-Geometric Approach to Nonlinear Integrable Equations. Springer, Berlin (1994)
3. Biachini, S., Bressan, A.: Vanishing viscosity solutions of nonlinear hyperbolic systems. Ann. Math. (2) **61**, 223–342 (2005)
4. Bressan, A.: Hyperbolic Systems of Conservation Laws. Oxford Univ. Press, Oxford (2000)
5. Bullough, R.K., Caudrey, P.J.: Solitons and the Korteweg–de Vries equation: Integrable systems in 1834–1995. Acta Appl. Math. **39**, 193–228 (1995)
6. Chen, G.-Q.: Convergence of the Lax–Friedrichs scheme for isentropic gas dynamics. III. Acta Math. Sci. **6**, 75–120 (1986)
7. Chen, G.-Q., Liu, J.-G.: Convergence of difference schemes with high resolution for conservation laws. Math. Comput. **66**, 1027–1053 (1997)
8. Colin de Verdiére, Y.: Pseudo-Laplacians. Ann. Inst. Fourier (Grenoble) **32**, 275–286 (1982)
9. Conway, E., Smoller, J.: Global solutions of the Cauchy problem for quasilinear first-order equations in several space variables. Commun. Pure Appl. Math. **19**, 95–105 (1966)
10. Courant, R., Friedrichs, K.O.: Supersonic Flow and Shock Waves. Springer, New York (1976). (First published in 1948)
11. Dafermos, C.M.: Hyperbolic Conservation Laws in Continuum Physics, 2nd edn. Springer, New York (2002)
12. Deift, P., McLaughlin, K.T.-R.: A continuum limit of the Toda lattice. Mem. Am. Math. Soc. **624**, x+216 (1998)
13. Deift, P., Venakides, S., Zhao, X.: New results in small dispersion KdV by an extension of the steepest descent method for Riemann–Hilbert problems. Internat. Math. Res. Notices **1997**, 286–299 (1997)
14. Deift, P., Zhao, X.: A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation. Ann. Math. (2) **137**, 294–368 (1993)
15. Delsarte, J.: Sur le gitter fuchsien. C. R. Acad. Sci. Paris **214**, 147–179 (1942)
16. Ding, X.X., Chen, G.-Q., Luo, P.Z.: Convergence of the Lax–Friedrichs scheme for isentropic gas dynamics. I. Acta Math. Sci. **5**, 415–432 (1985)
17. Ding, X.X., Chen, G.-Q., Luo, P.Z.: Convergence of the Lax–Friedrichs scheme for isentropic gas dynamics. II. Acta Math. Sci. **5**, 433–472 (1985)
18. Ding, X.X., Chen, G.-Q., Luo, P.Z.: A supplement to the papers: "Convergence of the Lax–Friedrichs scheme for isentropic gas dynamics. II, III". Acta Math. Sci. **9**, 43–44 (1989)
19. Dubrovin, B.A.: Inverse problem for periodic finite-zoned potentials in the theory of scattering. Funct. Anal. Appl. **9**, 61–62 (1975)
20. Dubrovin, B.A.: Periodic problems for the Korteweg–de Vries equation in the class of finite band potentials. Funct. Anal. Appl. **9**, 215–223 (1975)

21. Dubrovin, B.A., Novikov, S.P.: A periodicity problem for the Korteweg–de Vries and Sturm–Liouville equations. Their connection with algebraic geometry. Dokl. Akad. Nauk SSSR **15**, 1597–1601 (1974)
22. Dubrovin, B.A., Novikov, S.P.: Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg–de Vries equation. Sov. Phys. JETP **40**, 1058–1063 (1975)
23. Duistermaat, J.J.: Fourier Integral Operators. Birkhäuser, Basel (1996)
24. Duistermaat, J.J., Guillemin, V.W.: The spectrum of positive elliptic operators and periodic bicharacteristics. Invent. Math. **29**, 39–79 (1975)
25. Duke, W., Rudnick, Z., Sarnak, P.: Density of integer points on affine homogeneous varieties. Duke Math. J. **71**, 143–179 (1993)
26. Evans, L.C.: Partial Differential Equations. Am. Math. Soc., Providence (1998)
27. Faddeev, L., Pavlov, B.: Scattering theory and automorphic functions. Proc. Steklov Inst. Math. **27**, 161–193 (1972)
28. Gorodnik, A., Oh, H., Shah, N.: Integral points on symmetric varieties and Satake compatifications: Preprint, arXiv:math/0610497v2 (2008)
29. Gardner, C.S.: Korteweg–de Vries equation and generalizations. IV. The Korteweg–de Vries equation as a Hamiltonian system. J. Math. Phys. **12**, 1548–1551 (1971)
30. Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Method for solving the Korteweg–de Vries equation. Phys. Rev. Lett. **19**, 1095–1097 (1967)
31. Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Korteweg–de Vries equation and generalizations. VI. Methods for exact solution. Commun. Pure Appl. Math. **27**, 97–133 (1974)
32. Gel'fand, I.M.: Some problems in the theory of quasilinear equations. Am. Math. Soc. Transl. **29**, 295–381 (1963)
33. Gesztesy, F., Holden, H.: Soliton Equations and Their Algebro-Geometric Solutions. $(1+1)$-Dimensional Continuous Models, vol. I. Cambridge Univ. Press, Cambridge (2003)
34. Glimm, J.: Solutions in the large for nonlinear hyperbolic systems of equations. Commun. Pure Appl. Math. **18**, 697–715 (1965)
35. Holden, H., Risebro, N.H.: Front Tracking for Hyperbolic Conservation Laws. Springer, New York (2007). Corrected 2nd printing
36. Hopf, E.: The partial differential equation $u_t + uu_x = \mu u_{xx}$. Commun. Pure Appl. Math. **3**, 201–230 (1950)
37. Hörmander, L.: Pseudo-differential operators and non-elliptic boundary value problems. Ann. Math. **83**, 129–209 (1966)
38. Hörmander, L.: The spectrum of a positive elliptic operator. Acta Math. **121**, 193–218 (1968)
39. Hörmander, L.: Fourier integral operators. I. Acta Math. **127**, 79–183 (1971)
40. Ikawa, M.: On the poles of the scattering matrix for two strictly convex obstacles. J. Math. Kyoto Univ. **23**, 127–194 (1983). Addendum, *loc. sit.* **23**, 795–802 (1983)
41. Its, A.R., Matveev, V.B.: Hill's operator with finitely many gaps. Funct. Anal. Appl. **9**, 65–66 (1975)
42. Its, A.R., Matveev, V.B.: Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg–de Vries equation. Theoret. Math. Phys. **23**, 343–355 (1975)
43. Jin, S., Levermore, C.D., McLaughlin, D.W.: The behavior of solutions of the NLS equation in the semiclassical limit. In: Ercolani, N.M., Gabitov, I.R., Levermore, C.D., Serre, D. (eds.) Singular Limits of Dispersive Waves. NATO Adv. Sci. Inst. Ser. B, Phys., vol. 320, pp. 235–255. Plenum, New York (1994)
44. Klainerman, S., Majda, A.: Formation of singularities for wave equations including the nonlinear vibrating string. Commun. Pure Appl. Math. **33**, 241–263 (1980)
45. Kružkov, S.N.: First order quasi-linear equations in several independent variables. Math. USSR Sb. **10**, 217–243 (1970)
46. LeVeque, R.J.: Finite Volume Methods for Hyperbolic Problems. Cambridge Univ. Press, Cambridge (2002)
47. Ludwig, D., Morawetz, C.: The generalized Huygens' principle for reflecting bodies. Commun. Pure Appl. Math. **22**, 189–205 (1969)

48. Melrose, R.: Singularities and energy decay in acoustical scattering. Duke Math. J. **46**, 43–59 (1979)
49. Melrose, R.: Polynomial bound on the number of scattering poles. J. Funct. Anal. **53**, 287–303 (1983)
50. McKean, H.P., van Moerbeke, P.: The spectrum of Hill's equation. Invent. Math. **30**, 217–274 (1975)
51. Müller, W.: The trace class conjecture in the theory of automorphic forms. Ann. Math. **130**, 473–529 (1989)
52. Novikov, S.P.: The periodic problem for the Korteweg–de Vries equation. Funct. Anal. Appl. **8**, 236–246 (1974)
53. Novikov, S.P.: A method for solving the periodic problem for the KdV equation and its generalizations. Rocky Mountain J. Math. **8**, 83–93 (1978)
54. Novikov, S.P., Manakov, S.V., Pitaevskii, L.P., Zakharov, V.E.: Theory of Solitons. Consultants Bureau, New York (1984)
55. Oleĭnik, O.A.: Discontinuous solutions of non-linear differential equations. Am. Math. Soc. Transl. Ser. **26**, 95–172 (1963)
56. Oleĭnik, O.A.: Uniqueness and stability of the generalized solution of the Cauchy problem for a quasi-linear equation. Am. Math. Soc. Transl. Ser. **33**, 285–290 (1963)
57. Patterson, S.: The Laplacian operator on a Riemann surface. I. Compos. Math. **32**, 83–107 (1975)
58. Patterson, S.: The Laplacian operator on a Riemann surface. II. Compos. Math. **32**, 71–112 (1976)
59. Patterson, S.: The Laplacian operator on a Riemann surface. III. Compos. Math. **33**, 227–259 (1976)
60. Patterson, S.J., Perry, P.A.: The divisor of Selberg's zeta function for Kleinian groups. Duke Math. J. **106**, 321–390 (2001)
61. Pego, R.: Origin of the KdV equation. Not. Am. Math. Soc. **45**, 358 (1998)
62. Phillips, R., Sarnak, P.: Perturbation theory for the Laplacian on automorphic functions. J. Am. Math. Soc. **5**, 1–32 (1992)
63. Ralston, J.: Solutions of the wave equation with localized energy. Commun. Pure Appl. Math. **22**, 807–823 (1969)
64. Riemann, G.F.B.: Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. Abh. König. Gesell. Wiss. Göttingen **8**, 43–65 (1860)
65. Selberg, A.: Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. J. Indian Math. Soc. **20**, 47–87 (1956)
66. Serre, D.: Systems of Conservation Laws, vol. 1. Cambridge Univ. Press, Cambridge (1999)
67. Smoller, J.: Shock Waves and Reaction–Diffusion Equations, 2nd edn. Springer, New York (1994)
68. Sogge, C., Zelditch, S.: Riemannian manifolds with maximal eigenfunction growth. Duke Math. J. **114**, 387–437 (2002)
69. Venakides, S.: The zero dispersion limit of the Korteweg–de Vries equation for initials with nontrivial reflection coefficient. Commun. Pure Appl. Math. **38**, 125–155 (1985)
70. Venakides, S.: The zero dispersion limit of the Korteweg–de Vries equation for initials with periodic initial data. Trans. Am. Math. Soc. **301**, 189–226 (1987)
71. Venakides, S.: The continuum limit of theta functions. Commun. Pure Appl. Math. **42**, 711–728 (1989)
72. Venakides, S., Deift, P., Oba, R.: The Toda shock problem. Commun. Pure Appl. Math. **44**, 1171–1242 (1991)
73. Zabusky, N.J., Kruskal, M.D.: Interaction of "solitons" in a collisionless plasma and the recurrence of initial states. Phys. Rev. Lett. **15**, 240–243 (1965)
74. Zakharov, V.E., Faddeev, L.D.: Korteweg–de Vries equation: A completely integrable Hamiltonian system. Funct. Anal. Appl. **5**, 280–287 (1971)
75. Zakharov, V.E., Shabat, A.B.: Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Sov. Phys. JETP **34**, 62–69 (1972)

76. Zakharov, V.E., Shabat, A.B.: A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I. Funct. Anal. Appl. **8**, 226–235 (1974)
77. Zakharov, V.E., Shabat, A.B.: Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II. Funct. Anal. Appl. **13**, 166–174 (1979)
78. Zworski, M.: Sharp polynomial bounds on the number of scattering poles for radial potentials. J. Funct. Anal. **82**, 370–403 (1989)

List of Publications for Peter D. Lax

1944

[1] Proof of a conjecture of P. Erdős on the derivative of a polynomial. *Bull. Amer. Math. Soc.*, 50:509–513.

1948

[2] The quotient of exponential polynomials. *Duke Math. J.*, 15:967–970.

1949

[3] (with R. Courant). On nonlinear partial differential equations with two independent variables. *Comm. Pure Appl. Math.*, 2:255–273.

1950

[4] (with R. Courant). Method of characteristics for the solution of nonlinear partial differential equations. In *Symposium on theoretical compressible flow, 29 June 1949*, Rep. NOLR-1132, pages 61–71. Naval Ordnance Laboratory, White Oak, Md.

[5] *Partial Differential Equations*. Lecture Notes. New York University.

1951

[6] A remark on the method of orthogonal projections. *Comm. Pure Appl. Math.*, 4:457–464.

1952

[7] On the existence of Green's function. *Proc. Amer. Math. Soc.*, 3:526–531.

[8] (with P.W. Berg). Fourth order operators. *Univ. e Politecnico Torino. Rend. Sem. Mat.*, 11:343–358.

1953

[9] Nonlinear hyperbolic equations. *Comm. Pure Appl. Math.*, 6:231–258.

H. Holden, R. Piene (eds.), *The Abel Prize*,
DOI 10.1007/978-3-642-01373-7_16,

1954

[10] Weak solutions of nonlinear hyperbolic equations and their numerical computation. *Comm. Pure Appl. Math.*, 7:159–193.
[11] (with A.N. Milgram). Parabolic equations. In *Contributions to the theory of partial differential equations*, Annals of Mathematics Studies, no. 33, pages 167–190. Princeton University Press, Princeton, NJ.
[12] The initial value problem for nonlinear hyperbolic equations in two independent variables. In *Contributions to the theory of partial differential equations*, Annals of Mathematics Studies, no. 33, pages 211–229. Princeton University Press, Princeton, NJ.
[13] Symmetrizable linear transformations. *Comm. Pure Appl. Math.*, 7:633–647.

1955

[14] Reciprocal extremal problems in function theory. *Comm. Pure Appl. Math.*, 8:437–453.
[15] (with R. Courant). Cauchy's problem for nonlinear hyperbolic differential equations in two independent variables. *Ann. Mat. Pura Appl. (4)*, 40:161–166.
[16] On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations. *Comm. Pure Appl. Math.*, 8:615–633.

1956

[17] (with R.D. Richtmyer). Survey of the stability of linear finite difference equations. *Comm. Pure Appl. Math.*, 9:267–293.
[18] (with R. Courant). The propagation of discontinuities in wave motion. *Proc. Nat. Acad. Sci. U.S.A.*, 42:872–876.
[19] A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations. *Comm. Pure Appl. Math.*, 9:747–766.

1957

[20] Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math.*, 10:537–566.
[21] Remarks on the preceding paper. *Comm. Pure Appl. Math.*, 10:617–622.
[22] A Phragmén–Lindelöf theorem in harmonic analysis and its application to some questions in the theory of elliptic equations. *Comm. Pure Appl. Math.*, 10:361–389.
[23] Asymptotic solutions of oscillatory initial value problems. *Duke Math. J.*, 24:627–646.

1958

[24] Differential equations, difference equations and matrix theory. *Comm. Pure Appl. Math.*, 11:175–194.

1959

[25] Translation invariant spaces. *Acta Math.*, 101:163–178.
[26] On difference schemes for solving initial value problems for conservation laws. In *Symposium on questions of numerical analysis: Proceedings of the Rome Symposium (30 June – 1 July 1958) organized by the Provisional International Computation Centre*, pages 69–78, Rome. Libreria Eredi Virgilio Veschi.

1960

[27] The scope of the energy method. *Bull. Amer. Math. Soc.*, 66:32–35.
[28] (with R.S. Phillips). Local boundary conditions for dissipative symmetric linear differential operators. *Comm. Pure Appl. Math.*, 13:427–455.
[29] (with B. Wendroff). Systems of conservation laws. *Comm. Pure Appl. Math.*, 13:217–237.

1961

[30] Translation invariant spaces. In *Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960)*, pages 299–306. Jerusalem Academic Press, Jerusalem.

[31] On the stability of difference approximations to solutions of hyperbolic equations with variable coefficients. *Comm. Pure Appl. Math.*, 14:497–520.

1962

[32] (with R.S. Phillips). The wave equation in exterior domains. *Bull. Amer. Math. Soc.*, 68:47–49.

[33] (with C.S. Morawetz and R.S. Phillips). The exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle. *Bull. Amer. Math. Soc.*, 68:593–595.

[34] A procedure for obtaining upper bounds for the eigenvalues of a Hermitian symmetric operator. In *Studies in mathematical analysis and related topics*, pages 199–201. Stanford Univ. Press, Stanford, CA.

[35] (with B. Wendroff). On the stability of difference schemes. *Comm. Pure Appl. Math.*, 15:363–371.

1963

[36] Nonlinear hyperbolic systems of conservation laws. In *Nonlinear Problems (Proc. Sympos., Madison, WI, 1962)*, pages 3–12. Univ. of Wisconsin Press, Madison, WI.

[37] (with C.S. Morawetz and R.S. Phillips). Exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle. *Comm. Pure Appl. Math.*, 16:477–486.

[38] An inequality for functions of exponential type. *Comm. Pure Appl. Math.*, 16:241–246.

[39] On the regularity of spectral densities. *Theory of Probability and its Applications*, 8(3):316–319. Also in *Teor. Verojatnost. i Primenen.*, 8:337–340 (in Russian).

[40] Survey of stability of different schemes for solving initial value problems for hyperbolic equations. In *Proc. Sympos. Appl. Math.*, volume XV, pages 251–258. Amer. Math. Soc., Providence, RI.

1964

[41] Development of singularities of solutions of nonlinear hyperbolic partial differential equations. *J. Mathematical Phys.*, 5:611–613.

[42] (with R.S. Phillips). Scattering theory. *Bull. Amer. Math. Soc.*, 70:130–142.

[43] (with B.Wendroff). Difference schemes for hyperbolic equations with high order of accuracy. *Comm. Pure Appl. Math.*, 17:381–398.

1965

[44] (with K.O. Friedrichs). Boundary value problems for first order operators. *Comm. Pure Appl. Math.*, 18:355–388.

[45] (with J.F. Adams and R.S. Phillips). On matrices whose real linear combinations are nonsingular. *Proc. Amer. Math. Soc.*, 16:318–322. Corrections, *Proc. Amer. Math. Soc.*, 17:945–947, 1966.

[46] Numerical solution of partial differential equations. *Amer. Math. Monthly*, 72(2, part II):74–84.

1966

[47] Stability of linear and nonlinear difference schemes. In *Numerical Solution of Partial Differential Equations (Proc. Sympos. Univ. Maryland, 1965)*, pages 193–196. Academic Press, New York.

[48] (with L. Nirenberg). On stability for difference schemes: A sharp form of Gårding's inequality. *Comm. Pure Appl. Math.*, 19:473–492.

[49] (with R.S. Phillips). Analytic properties of the Schrödinger scattering matrix. In *Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U. S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, WI, 1965)*, pages 243–253. Wiley, New York.

[50] (with J.-P. Auffray). Aspects mathématiques de la Mécanique de phase. *C. R. Acad. Sc. Paris. Série B*, 263:1355–1357.

[51] Scattering theory; Remarks on the energy method; On scattering theory for a geometrical optics model. In *Proc. Conf. on Dispersion Theory, Cambridge, MA*, pages 36–42. M11.

1967

[52] (with J. Glimm). Decay of solutions of systems of hyperbolic conservation laws. *Bull. Amer. Math. Soc.*, 73:105.

[53] (with R.S. Phillips). *Scattering theory*. Academic Press, New York. Second edition, Academic Press, Boston, MA, 1989.

[54] (with R.S. Phillips). The acoustic equation with an indefinite energy form and the Schrödinger equation. *J. Functional Analysis*, 1:37–83.

[55] Hyperbolic difference equations: A review of the Courant–Friedrichs–Lewy paper in the light of recent developments. *IBM J. Res. Develop.*, 11:235–238.

[56] (with R.S. Phillips). Scattering theory for transport phenomena. In *Functional Analysis (Proc. Conf., Irvine, CA, 1966)*, pages 119–130. Academic Press, London.

[57] (with L. Nirenberg). A sharp inequality for pseudo-differential and difference operators. In *Singular Integrals (Proc. Sympos. Pure Math., Chicago, IL, 1966)*, pages 213–217. Amer. Math. Soc., Providence, R.I.

[58] (with K.O. Friedrichs). On symmetrizable differential operators. In *Singular Integrals (Proc. Sympos. Pure Math., Chicago, IL, 1966)*, pages 128–137. Amer. Math. Soc., Providence, R.I.

[59] II. The Mathematics Curriculum. *SIAM Review*, 9(2):312–325.

1968

[60] Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490.

[61] (with R.S. Phillips). Scattering theory. In *Proc. Internat. Congr. Math. (Moscow, 1966)*, pages 542–545. Izdat. "Mir", Moscow.

1969

[62] Nonlinear partial differential equations and computing. *SIAM Rev.*, 11:7–19.

[63] (with R.S. Phillips). Decaying modes for the wave equation in the exterior of an obstacle. *Comm. Pure Appl. Math.*, 22:737–787. Also in *Izdat. "Mir", Moscow*, 1971 (in Russian).

[64] Toeplitz operators. In A.K. Aziz, editor, *Lecture Series in Differential Equations*, volume II, pages 257–282. Van Nostrand, New York.

1970

[65] (with R.S. Phillips). Purely decaying modes for the wave equation in the exterior of an obstacle. In *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*, pages 11–20. Univ. of Tokyo Press, Tokyo.

[66] (with J. Glimm). *Decay of solutions of systems of nonlinear hyperbolic conservation laws*. Number 101 in Memoirs of the American Mathematical Society. American Mathematical Society, Providence, RI.

[67] (with R.S. Phillips). The eigenvalues of the Laplace operator for the exterior problem. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, CA, 1968)*, pages 147–148. Amer. Math. Soc., Providence, RI.

[68] Invariant functionals of nonlinear equations of evolution. In *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*, pages 240–251. Univ. of Tokyo Press, Tokyo.
[69] (with R.S. Phillips). The Paley–Wiener theorem for the Radon transform. *Comm. Pure Appl. Math.*, 23:409–424. A correction, *Comm. Pure Appl. Math.*, 24:279–288, 1971.

1971

[70] Approximation of measure preserving transformations. *Comm. Pure Appl. Math.*, 24:133–135.
[71] H. Brezis, W. Rosenkrantz, and B. Singer. On a degenerate elliptic–parabolic equation occurring in the theory of probability. *Comm. Pure Appl. Math.*, 24:395–416. Appendix by P.D. Lax.
[72] (with K.O. Friedrichs). Systems of conservation equations with a convex extension. *Proc. Nat. Acad. Sci. U.S.A.*, 68:1686–1688.
[73] (with R.S. Phillips). A logarithmic bound on the location of the poles of the scattering matrix. *Arch. Rational Mech. Anal.*, 40:268–280.
[74] The formation and decay of shock waves. In *Visiting scholars' lectures (Texas Tech Univ., Lubbock, TX, 1970/71)*, pages 107–139. Math. Ser., No. 9. Texas Tech Press, Texas Tech. Univ., Lubbock, TX.
[75] Shock waves and entropy. In *Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, WI, 1971)*, pages 603–634. Academic Press, New York.
[76] (with R.S. Phillips). Scattering theory. *Rocky Mountain J. Math.*, 1(1):173–223.
[77] Nonlinear partial differential equations of evolution. In *Actes, Congrès Intern. Math., Nice, 1970, Tome 2*, pages 831–840. Gauthier–Villars, Paris.

1972

[78] (with R.S. Phillips). On the scattering frequencies of the Laplace operator for exterior domains. *Comm. Pure Appl. Math.*, 25:85–101.
[79] The formation and decay of shock waves. *Amer. Math. Monthly*, 79:227–241.
[80] (with R.S. Phillips). Scattering theory for the acoustic equation in an even number of space dimensions. *Indiana Univ. Math. J.*, 22:101–134.
[81] Exponential modes of the linear Boltzmann equation. In F.A. Grunbaum, editor, *Boltzmann Equation Seminar*, pages 111–123. Courant Institute, New York University.

1973

[82] The differentiability of Pólya's function. *Advances in Math.*, 10:456–464.
[83] *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*. Society for Industrial and Applied Mathematics, Philadelphia, PA.
[84] (with R.S. Phillips). Scattering theory for dissipative hyperbolic systems. *J. Functional Analysis*, 14:172–235.

1974

[85] Periodic solutions of the KdV equations. In *Nonlinear wave motion (Proc. AMS–SIAM Summer Sem., Clarkson Coll. Tech., Potsdam, NY, 1972)*, volume 15 of *Lectures in Appl. Math.*, pages 85–96. Amer. Math. Soc., Providence, RI.
[86] Invariant functionals of nonlinear equations of evolution. In S. Leibovich and A.R. Seebass, editors, *Nonlinear waves*, pages 291–310. Cornell University Press, Ithaca, NY.
[87] (with R.S. Phillips). A scattering theory for automorphic functions. In *Séminaire Goulaouic–Schwartz 1973–1974: Équations aux dérivées partielles et analyse fonctionnelle, Exp. No. 23*, page 7. Centre de Math., École Polytech., Paris.
[88] Applied mathematics and computing. In *Proceedings of Symposia in Applied Mathematics*, volume 20, pages 57–66, Providence, RI. Amer. Math. Soc.

1975

[89] Periodic solutions of the KdV equation. *Comm. Pure Appl. Math.*, 28:141–188.
[90] Almost periodic behavior of nonlinear waves. *Advances in Math.*, 16:368–379.

1976

[91] Almost periodic solutions of the KdV equation. *SIAM Rev.*, 18(3):351–375.
[92] (with A. Harten and J.M. Hyman). On finite-difference approximations and entropy conditions for shocks. *Comm. Pure Appl. Math.*, 29(3):297–322. With an appendix by B. Keyfitz.
[93] On the factorization of matrix-valued functions. *Comm. Pure Appl. Math.*, 29(6):683–688.
[94] (with R.S. Phillips). *Scattering theory for automorphic functions*. Number 87 in Annals of Mathematics Studies. Princeton Univ. Press, Princeton, NJ.
[95] Almost periodic behavior of nonlinear waves. In *Surveys in applied mathematics (Proc. First Los Alamos Sympos. Math. in Natural Sci., Los Alamos, NM, 1974)*, pages 259–270. Academic Press, New York.

1977

[96] (with R.S. Phillips). The scattering of sound waves by an obstacle. *Comm. Pure Appl. Math.*, 30(2):195–233.
[97] P.D. Lax, editor. *Mathematical aspects of production and distribution of energy*. American Mathematical Society, Providence, RI.
[98] A Hamiltonian approach to the KdV and other equations. In *Group theoretical methods in physics (Proc. Fifth Internat. Colloq., Univ. Montréal, Montreal, Que., 1976)*, pages 39–57. Academic Press, New York.
[99] The bomb, Sputnik, computers and European mathematics. In *Bicentennial Tribute to American Mathematics, San Antonio, 1976*, pages 129–135. Mathematical Association of America.

1978

[100] (with A. Lax). On sums of squares. *Linear Algebra and Appl.*, 20(1):71–75.
[101] (with R.S. Phillips). An example of Huygens' principle. *Comm. Pure Appl. Math.*, 31(4):415–421.
[102] (with M.S. Mock). The computation of discontinuous solutions of linear hyperbolic equations. *Comm. Pure Appl. Math.*, 31(4):423–430.
[103] (with R.S. Phillips). Scattering theory for domains with non-smooth boundaries. *Arch. Rational Mech. Anal.*, 68(2):93–98.
[104] A Hamiltonian approach to the KdV and other equations. In *Nonlinear evolution equations (Proc. Sympos., Univ. Wisconsin, Madison, WI, 1977)*, volume 40 of *Publ. Math. Res. Center Univ. Wisconsin*, pages 207–224. Academic Press, New York.
[105] Accuracy and resolution in the computation of solutions of linear and nonlinear equations. In *Recent advances in numerical analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, WI, 1978)*, volume 41 of *Publ. Math. Res. Center Univ. Wisconsin*, pages 107–117. Academic Press, New York.
[106] S.Z. Burstein, P.D. Lax, and G.A. Sod, editors. *Lectures on combustion theory*. New York University, Courant Mathematics and Computing Laboratory, New York.
[107] The numerical solution of the equations of fluid dynamics. In *Lectures on combustion theory (Courant Inst., New York Univ., New York, 1977)*, pages 1–60. New York Univ., New York.
[108] Chemical kinetics. In *Lectures on combustion theory (Courant Inst., New York Univ., New York, 1977)*, pages 122–136. New York Univ., New York.
[109] (with R.S. Phillips). The time delay operator and a related trace formula. In *Topics in functional analysis (essays dedicated to M.G. Kreĭn on the occasion of his 70th birthday)*, volume 3 of *Adv. in Math. Suppl. Stud.*, pages 197–215. Academic Press, New York.

1979

[110] (with R.S. Phillips). Translation representations for the solution of the non-Euclidean wave equation. *Comm. Pure Appl. Math.*, 32(5):617–667. Correction, *Comm. Pure Appl. Math.*, 33(5):685, 1980.

[111] Recent methods for computing discontinuous solutions—a review. In *Computing methods in applied sciences and engineering (Proc. Third Internat. Sympos., Versailles, 1977), II*, volume 91 of *Lecture Notes in Phys.*, pages 3–12. Springer, Berlin.

[112] (with C.D. Levermore). The zero dispersion limit for the Korteweg-de Vries KdV equation. *Proc. Nat. Acad. Sci. U.S.A.*, 76(8):3602–3606.

1980

[113] (with R.S. Phillips). Scattering theory for automorphic functions. *Bull. Amer. Math. Soc. (N.S.)*, 2(2):261–295.

[114] On the notion of hyperbolicity. *Comm. Pure Appl. Math.*, 33(3):395–397.

1981

[115] (with R.S. Phillips). Translation representations for the solution of the non-Euclidean wave equation. II. *Comm. Pure Appl. Math.*, 34(3):347–358.

[116] (with A. Harten). A random choice finite difference scheme for hyperbolic conservation laws. *SIAM J. Numer. Anal.*, 18(2):289–315.

[117] (with R.S. Phillips). The translation representation theorem. *Integral Equations Operator Theory*, 4(3):416–421.

[118] (with R.S. Phillips). A local Paley–Wiener theorem for the Radon transform in real hyperbolic spaces. In *Mathematical analysis and applications, Part B*, volume 7 of *Adv. in Math. Suppl. Stud.*, pages 483–487. Academic Press, New York.

[119] Applied mathematics, 1945 to 1975. In *American mathematical heritage: algebra and applied mathematics (El Paso, TX, 1975/Arlington, TX, 1976)*, volume 13 of *Math. Ser.*, pages 95–100. Texas Tech. Univ., Lubbock, TX.

[120] (with R.S. Phillips). The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces. In *Functional analysis and approximation (Oberwolfach, 1980)*, volume 60 of *Internat. Ser. Numer. Math.*, pages 373–383. Birkhäuser, Basel.

1982

[121] The multiplicity of eigenvalues. *Bull. Amer. Math. Soc. (N.S.)*, 6(2):213–214.

[122] (with R.S. Phillips). A local Paley–Wiener theorem for the Radon transform of L_2 functions in a non-Euclidean setting. *Comm. Pure Appl. Math.*, 35(4):531–554.

[123] (with R.S. Phillips). The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces. *J. Funct. Anal.*, 46(3):280–350.

[124] (with R.S. Phillips). The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces. In *Toeplitz centennial (Tel Aviv, 1981)*, volume 4 of *Operator Theory: Adv. Appl.*, pages 365–375. Birkhäuser, Basel.

[125] (with C.D. Levermore). The zero dispersion limit for the Korteweg–de Vries (KdV) equation. In *Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980)*, pages 717–729, Beijing. Science Press.

1983

[126] (with A. Harten and B. van Leer). On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. *SIAM Rev.*, 25(1):35–61.

[127] H. Hopf. *Differential geometry in the large*, volume 1000 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin. Notes taken by P. Lax and J. Gray. Second edition, 1989.

[128] (with C.D. Levermore). The small dispersion limit of the Korteweg–de Vries equation. I. *Comm. Pure Appl. Math.*, 36(3):253–290.

[129] (with C.D. Levermore). The small dispersion limit of the Korteweg–de Vries equation. II. *Comm. Pure Appl. Math.*, 36(5):571–593.

[130] (with C.D. Levermore). The small dispersion limit of the Korteweg–de Vries equation. III. *Comm. Pure Appl. Math.*, 36(6):809–829.

[131] H. Fujita, P.D. Lax, and G. Strang, editors. *Nonlinear partial differential equations in applied science*, volume 5 of *Lecture Notes in Numerical and Applied Analysis*, Tokyo. Kinokuniya Book Store Co. Ltd. North-Holland Mathematics Studies, 81.

[132] (with S.Z. Burstein and A. Lax). *Calculus with applications and computing. Vol. I.* Springer, Berlin.

1984

[133] A. Harten. On a class of high resolution total-variation-stable finite-difference schemes. *SIAM J. Numer. Anal.*, 21(1):1–23, 1984. With an appendix by P. D. Lax.

[134] (with R.S. Phillips). Translation representation for automorphic solutions of the wave equation in non-Euclidean spaces. I. *Comm. Pure Appl. Math.*, 37(3):303–328.

[135] (with R.S. Phillips). Translation representations for automorphic solutions of the wave equation in non-Euclidean spaces. II. *Comm. Pure Appl. Math.*, 37(6):779–813.

[136] Shock waves, increase of entropy and loss of information. In *Seminar on nonlinear partial differential equations (Berkeley, CA, 1983)*, volume 2 of *Math. Sci. Res. Inst. Publ.*, pages 129–171. Springer, New York.

[137] The zero dispersion limit for the KdV equation. In *Differential equations (Birmingham, AL, 1983)*, volume 92 of *North-Holland Math. Stud.*, pages 387–390. North-Holland, Amsterdam.

[138] Problems solved and unsolved concerning linear and nonlinear partial differential equations. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warszaw, 1983)*, pages 119–137, Warszaw. PWN.

[139] In praise of calculus... *The College Mathematical Journal*, 15:378–380.

1985

[140] (with R.S. Phillips). Translation representations for automorphic solutions of the wave equation in non-Euclidean spaces. III. *Comm. Pure Appl. Math.*, 38(2):179–207.

[141] (with R.S. Phillips). Translation representations for automorphic solutions of the wave equation in non-Euclidean spaces; the case of finite volume. *Trans. Amer. Math. Soc.*, 289(2):715–735.

[142] (with R.J. LeVeque and C.S. Peskin). Solution of a two-dimensional cochlea model using transform techniques. *SIAM J. Appl. Math.*, 45(3):450–464.

[143] (with P. Constantin and A. Majda). A simple one-dimensional model for the three-dimensional vorticity equation. *Comm. Pure Appl. Math.*, 38(6):715–724.

1986

[144] Hyperbolic systems of conservation laws in several space variables. In *Current topics in partial differential equations*, pages 327–341. Kinokuniya, Tokyo.

[145] On dispersive difference schemes. *Phys. D*, 18(1–3):250–254.

[146] Mathematics and computing. *J. Stat. Phys.*, 43(5–6):749–756.

[147] Mathematics and its applications. *Math. Intelligencer*, 8(4):14–17. Also available in *Pokroky Mat. Fyz. Astronom.*, 32(6):309–314, 1987 (in Czech).

[148] (with A. Jameson). Conditions for the construction of multipoint total variation diminishing difference schemes. *Appl. Numer. Math.*, 2(3–5):335–345. Corrigendum, *Appl. Numer. Math.*, 3(3):289, 1987.

[149] On the weak convergence of dispersive difference schemes. In *Oscillation theory, computation, and methods of compensated compactness (Minneapolis, MN, 1985)*, volume 2 of *IMA Vol. Math. Appl.*, pages 107–113. Springer, New York.

[150] K.O. Friedrichs. *Selecta. Vol. 1, 2*. Birkhäuser Boston Inc., Boston, MA. Edited by C.S. Morawetz. With commentaries by P. Lax and others.

1987

[151] On symmetrizing hyperbolic differential equations. In *Nonlinear hyperbolic problems (St. Etienne, 1986)*, volume 1270 of *Lecture Notes in Math.*, pages 150–151. Springer, Berlin.

[152] The soul of mathematics. In *Studies in Mathematics and Its Applications*, volume 18. Patterns and Waves, pages viii–ix. North-Holland, Amsterdam.

1988

[153] (with R.J. LeVeque and C.S. Peskin). Solution of a two-dimensional cochlea model with fluid viscosity. *SIAM J. Appl. Math.*, 48(1):191–213.

[154] Oscillatory solutions of partial differential and difference equations. In *Mathematics applied to science (New Orleans, LA, 1986)*, pages 155–170. Academic Press, Boston, MA.

[155] (with J. Goodman). On dispersive difference schemes. I. *Comm. Pure Appl. Math.*, 41(5):591–613.

[156] Mathematics and computing. In *ICIAM '87: Proceedings of the First International Conference on Industrial and Applied Mathematics (Paris, 1987)*, pages 137–143. SIAM, Philadelphia, PA.

1989

[157] The flowering of applied mathematics in America. In *A century of mathematics in America, Part II*, volume 2 of *Hist. Math.*, pages 455–466. Amer. Math. Soc., Providence, RI. Also in *SIAM Rev.*, 31(4):533–541.

[158] *The flowering of applied mathematics in America*. AMS–MAA Joint Lecture Series. American Mathematical Society, Providence, RI. A joint AMS–MAA invited address presented in Providence, Rhode Island, August 1988 (Video presentation).

[159] Science and computing. *Proceedings of the IEEE*, 77(7):1018–1019.

[160] Writing mathematics well. Review of L. Gillman. *American Math. Monthly*, 96:380–381.

[161] Remembering the strange attractions of Stan Ulam. *Physics Today*, 42(6):69–72. Also available in *Bull. Amer. Math. Soc.*, 22:304–310, 1990 and *St. Petersburg Math. J.*, 4:629–632, 1993.

[162] Deterministic turbulence. In *Symmetry in Nature. Volume in Honor of L. A. Radicati de Brozolo II*, pages 485–490. Scuola Normale Superiore.

1990

[163] The ergodic character of sequences of pedal triangles. *Amer. Math. Monthly*, 97(5):377–381.

[164] Remembering John von Neumann. In *The legacy of John von Neumann (Hempstead, NY, 1988)*, volume 50 of *Proc. Sympos. Pure Math.*, pages 5–7. Amer. Math. Soc., Providence, RI.

[165] On Riemann's paper: Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. In *Berhard Riemann. Gesammelte Mathematische Werke, Wissenschaftliche Nachlass und Nachträge*, pages 807–810. Springer, Berlin.

1991

[166] (with T.Y. Hou). Dispersive approximations in fluid dynamics. *Comm. Pure Appl. Math.*, 44(1):1–40.

[167] Deterministic theories of turbulence. In *Frontiers in pure and applied mathematics*, pages 179–184. North-Holland, Amsterdam.

[168] The zero dispersion limit, a deterministic analogue of turbulence. *Comm. Pure Appl. Math.*, 44(8–9):1047–1056.

1992

[169] (with R.S. Phillips). Translation representation for automorphic solutions of the wave equation in non-Euclidean spaces. IV. *Comm. Pure Appl. Math.*, 45(2):179–201.

1993

[170] The existence of eigenvalues of integral operators. *Indiana Univ. Math. J.*, 42(3):889–891.

[171] (with C.D. Levermore and S. Venakides). The generation and propagation of oscillations in dispersive initial value problems and their limiting behavior. In *Important developments in soliton theory*, Springer Ser. Nonlinear Dynam., pages 205–241. Springer, Berlin.

1994

[172] Trace formulas for the Schroedinger operator. *Comm. Pure Appl. Math.*, 47(4):503–512.

[173] Cornelius Lanczos (1893–1974), and the Hungarian phenomenon in science and mathematics. In *Proceedings of the Cornelius Lanczos International Centenary Conference (Raleigh, NC, 1993)*, pages xlix–lii. SIAM, Philadelphia, PA.

1995

[174] A short path to the shortest path. *Amer. Math. Monthly*, 102(2):158–159.

[175] L.A. Ying. *Infinite element methods*. Peking University Press, Beijing, Chinese edition. With a preface by P.D. Lax.

[176] Ami Harten [?–1994]. *Comm. Pure Appl. Math.*, 48(12):1303. Erratum, *Comm. Pure Appl. Math.*, 49(8):867, 1996.

[177] Computational fluid dynamics at the Courant Institute. In M. Hafez and K. Oshima, editors, *Computational Fluid Dynamics Review 1995*, pages 1–5. Wiley, New York.

1996

[178] (with X.-D. Liu). Positive schemes for solving multi-dimensional hyperbolic systems of conservation laws. In *Proceedings of the VIII International Conference on Waves and Stability in Continuous Media, Part I (Palermo, 1995)*, number 45, part I, pages 367–375. Circolo Matematico di Palermo, Palermo.

[179] Outline of a theory of the KdV equation. In *Recent mathematical methods in nonlinear wave propagation*, volume 1640 of *Lecture Notes in Mathematics*, pages 70–102. Springer-Verlag, Berlin. Edited by T. Ruggeri.

[180] (with X.-D. Liu). Positive schemes for solving multidimensional hyperbolic conservation laws. *Comp. Fluid Dynamics*, 5:133–156.

[181] The old days. In B.A. Case, editor, *A Century of Mathematical Meetings*, pages 257–282. Amer. Math. Soc., Providence, RI.

1997

[182] *Linear algebra*. John Wiley & Sons Inc., New York. Second edition, 2007.

[183] The zero dispersion limit, a deterministic analogue of turbulence. In *Nonlinear evolutionary partial differential equations (Beijing, 1993)*, volume 3 of *AMS/IP Stud. Adv. Math.*, pages 53–64. Amer. Math. Soc., Providence, RI, 1997.

[184] A. Harten. High resolution schemes for hyperbolic conservation laws. *J. Comput. Phys.*, 135(2):259–278. With an introduction by P. Lax. (Commemoration of the 30th anniversary of J. Comput. Phys.)

1998

[185] (with X.-D. Liu). Solution of two-dimensional Riemann problems of gas dynamics by positive schemes. *SIAM J. Sci. Comput.*, 19(2):319–340 (electronic).

[186] On the discriminant of real symmetric matrices. *Comm. Pure Appl. Math.*, 51(11–12):1387–1396.

[187] (with A. Harten, C.D. Levermore, and W.J. Morokoff). Convex entropies and hyperbolicity for general Euler equations. *SIAM J. Numer. Anal.*, 35(6):2117–2127 (electronic).

[188] The beginnings of applied mathematics after the Second World War. *Quart. Appl. Math.*, 56(4):607–615.

[189] J. Leray. *Selected papers. Œuvres scientifiques. Vol. II*. Springer-Verlag, Berlin. With an introduction by P.D. Lax. Edited by P. Malliavin.

1999

[190] Change of variables in multiple integrals. *Amer. Math. Monthly*, 106(6):497–501.

[191] (with S. Hildebrandt). *Otto Toeplitz*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 319. Universität Bonn, Mathematisches Institut, Bonn.

[192] Mathematics and computing. In A.G. Bearn, editor, *Useful knowledge*, pages 23–44. American Philosophical Society.

[193] A mathematician who lived for mathematics. *Physics Today*, 52(4):69–70.

2000

[194] (with A. Borel and G.M. Henkin). Jean Leray (1906–1998). *Notices Amer. Math. Soc.*, 47(3):350–359.

[195] V. Arnold, M. Atiyah, P. Lax, and B. Mazur, editors. *Mathematics: frontiers and perspectives*. Amer. Math. Soc., Providence, RI.

[196] Mathematics and computing. In *Mathematics: frontiers and perspectives*, pages 417–432. Amer. Math. Soc., Providence, RI.

[197] C.D. Olds, A. Lax, and G.P. Davidoff. *The geometry of numbers*, volume 41 of *Anneli Lax New Mathematical Library*. Mathematical Association of America, Washington, DC. Appendix I by P.D. Lax.

[198] On the accuracy of Glimm's scheme. *Methods Appl. Anal.*, 7(3):473–477.

2001

[199] Change of variables in multiple integrals. II. *Amer. Math. Monthly*, 108(2):115–119.

[200] (with L. Carleson, S.-Y.A. Chang, P.W. Jones, M. Keel, N. Makarov, D. Sarason, W. Schlag, and B. Simon). Thomas H. Wolff (1954–2000). *Notices Amer. Math. Soc.*, 48(5):482–490.

[201] The Radon transform and translation representation. *J. Evol. Equ.*, 1(3):311–323.

[202] (with E. Magenes and R. Temam). Jacques-Louis Lions (1928–2001). *Notices Amer. Math. Soc.*, 48(11):1315–1321.

[203] Short mathematical essays. In S.S. Chern and F. Hirzebruch, editors, *Wolf Prize in Mathematics*, volume 2, pages 221–239. World Scientific, Singapore.

2002

[204] H. Lewy. *Hans Lewy Selecta. Vol. 1*. Contemporary Mathematicians. Birkhäuser Boston Inc., Boston, MA. Edited by D. Kinderlehrer. Commentaries by P. Lax (see [55]) and others.

[205] *Functional analysis*. Wiley-Interscience, New York.

[206] Jürgen Moser, 1928–1999. *Ergodic Theory Dynam. Systems*, 22(5):1337–1342.

2003

[207] (with W. Jäger and C.S. Morawetz). Olga Arsen'evna Oleĭnik (1925–2001). *Notices Amer. Math. Soc.*, 50(2):220–223. Also in *Tr. Semin. im. I. G. Petrovskogo*, (23):7–15 (in Russian).

[208] (with X.-D. Liu). Positive schemes for solving multi-dimensional hyperbolic systems of conservation laws. II. *J. Comput. Phys.*, 187(2):428–440.

[209] Max Shiffman (1914–2000). *Notices Amer. Math. Soc.*, 50(11):1401.

[210] From the early days of the theory of distributions. *Gaz. Math.* (98, suppl.):63–64.

[211] Richard Courant (1888–1972). *Bibliographical Memoirs. National Academy of Sciences*, 82:78–97.

[212] (with G. Francsics). A fundamental domain for the Picard modular group in C^2. The Erwin Schrödinger International Institute for Mathematical Physics, Vienna, Austria, preprint 1273 (2003), 19 pp.

2004

[213] (with S. Friedlander, C. Morawetz, L. Nirenberg, G. Seregin, N. Ural'tseva, and M. Vishik). Olga Alexandrovna Ladyzhenskaya (1922–2004). *Notices Amer. Math. Soc.*, 51(11):1320–1331.

2005

[214] (with G. Francsics). A semi-explicit fundamental domain for a Picard modular group in complex hyperbolic space. In *Geometric analysis of PDE and several complex variables*, volume 368 of *Contemp. Math.*, pages 211–226. Amer. Math. Soc., Providence, RI.

[215] *Selected papers. Vol. I*. Springer, New York. Edited by P. Sarnak and A. Majda.

[216] *Selected papers. Vol. II*. Springer, New York. Edited by P. Sarnak and A. Majda.

[217] *John von Neumann: selected letters*, volume 27 of *History of Mathematics*. American Mathematical Society, Providence, RI. With a foreword by P. Lax. Edited by M. Rédei.

[218] (with G. Francsics). An explicit fundamental domain for the Picard modular group in two complex dimensions. Preprint, arXiv:math.CV/0509708 v1 29 Sep 2005.

[219] John von Neumann: The early years, the years at Los Alamos, and the road to computing. *SIAM News*, 38(2):9–10.

2006

[220] (with G. Francsics). Analysis of a Picard modular group. *Proc. Natl. Acad. Sci. USA*, 103(30):11103–11105 (electronic).

[221] Gibbs phenomena. *J. Sci. Comput.*, 28(2–3):445–449.

[222] *Hyperbolic partial differential equations*, volume 14 of *Courant Lecture Notes in Mathematics*. Amer. Math. Soc., Providence, RI.

[223] Thoughts on Gaetano Fichera. *Rend. Accad. Naz. XL Mem. Math. Appl.* (5) 30:1.

2007

[224] Computational fluid dynamics. *J. Sci. Comp.*, 31(1–2):185–193.
[225] The Cauchy integral theorem. *Amer. Math. Monthly*, 114(8):725–727.

2008

[226] Mathematics and physics. *Bull. Amer. Math. Soc. (N.S.)*, 45:135–152.
[227] A curious functional equation. *J. Anal. Math.* 105:383–390.

Curriculum Vitae for Peter David Lax

Born:	May 1, 1926, Budapest, Hungary
Degrees/education:	AB New York University 1947 PhD New York University 1949
Positions:	Staff Member, Los Alamos Scientific Laboratory, 1945–46 Staff Member, Los Alamos Scientific Laboratory, 1950 Assistant Professor, New York University, 1951–58 Professor, New York University, 1958–
Visiting positions:	Fulbright Lecturer, Germany 1958
Memberships:	Academies des Sciences, Paris 1982 National Academy of Sciences, USA 1982 American Academy of Arts and Sciences, USA 1982 New York Academy of Sciences, 1982 Russian Academy of Sciences, 1989 Hungarian Academy of Sciences, 1993 Academy Sinica, Beijing 1993 Moscow Mathematical Society, 1995 London Mathematical Society, 1997 Norwegian Academy of Science and Letters, 2009
Awards and prizes:	Lester R Ford Award (1966 and 1973) von Neumann Lecturer, SIAM 1960 Chauvenet Prize, 1974 Norbert Wiener Prize, 1975

Semmelweis medal, 1976
Award in Applied Mathematics from the National Academy of Sciences, 1983
National Medal of Science, 1986
Wolf Prize, 1987
Leroy Steele Prize, 1992
Abel Prize, 2005
SIAM Prize for Distinguished Service, 2006

Honorary degrees:
Kent State University, 1975
University of Paris, 1979
RWTH Aachen, 1988
Heriot-Watt University, 1990
Tel Aviv University, 1992
University of Maryland, Baltimore 1993
Brown University, 1993
Beijing University, 1993
Texas A & M University, 2000
New Jersey Institute of Technology, 2007
Lund University, 2008

Presidencies:
American Mathematical Society, 1977–80
Director, Courant Institute, 1972–80