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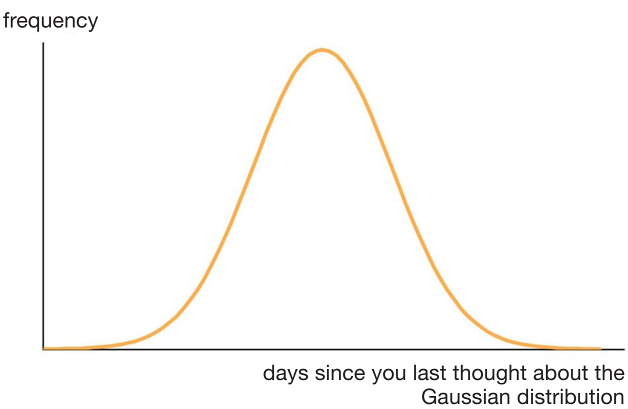
A Glimpse of the Laureate’s Work

By Matt Parker

Michel Talagrand is an expert at understanding and taming complicated random processes. Randomness can arise in a wide range of ways, and Talagrand has explored many different types. One of the most common, and arguably most important, types of randomness arises from “Gaussian processes”. The Gaussian distribution has been a constant feature of Talagrand’s career so it’s worth considering it as a tangible example when exploring his work.

A Gaussian distribution (sometimes called the “normal distribution” or “bell curve”) occurs with surprising frequency in the world around us. The mass of babies at birth, the test results students get at school and the ages athletes retire at are all seemingly random things which neatly follow the Gaussian distribution. These are characterised by having an average in the middle which most values are close to, and then decreasing numbers of cases as values move further above and below the average.



When observing a random process there are some things which it could be nice to know. For example, if you take the average of the values being produced, how close is that likely to be to the true average value of the underlying Gaussian process? How big or small are the possible values in the future likely to be?

Talagrand produced rigorous and tight thresholds with specific uncertainties so when these kind of random, stochastic processes are observed we know exactly how confident we can reasonably be about what the process will do going forward.

Viewing Talagrand’s work through the example of a single Gaussian distribution is an extreme oversimplification of how general and wide ranging his results were. More complicated situations involve several different random variables, with different distributions that combine in complicated ways. Instead of being a simple distribution, if each random variable is considered to be an independent axis, the total probability space which results is a multi-dimensional object, beyond direct human comprehension.

Talagrand was able to probe and understand these higher dimensional spaces. Asking probabilities questions about these spaces is akin to trying to find their higher dimensional volumes, and area of mathematics called “measure theory”. In simple 2D space, we can understand area as the sum of the areas if we split a distribution up into tiny squares and then count all of them. In higher dimensions, it is not that simple to work out the measure of a set of points. The Banach–Tarski paradox is a famous result that if you are not careful: a ball can be split into pieces which actually total twice its original volume.

The challenge becomes finding spaces which are well behaved enough to have a meaningful measure, and then set bounds on what that measure can be. Talagrand was able to do this with spaces beyond what human intuition is capable of dealing with. These higher dimension shapes can behave in very surprising ways.

Talagrand's work converting these objects into probability insights often involve knowing where the hyper-volume was likely to be concentrated around. Which for something like a ball, it would seem obvious: a ball is defined as all the points within the radius distance of a central point. But that can be deceiving.

Imaging a 2D ball, a circular disk, trapped between four unit disks, surrounded by a box (see Figure 1). The biggest radius that central circle can possibly have is 0.41421, which is much smaller than the unit circles around it.

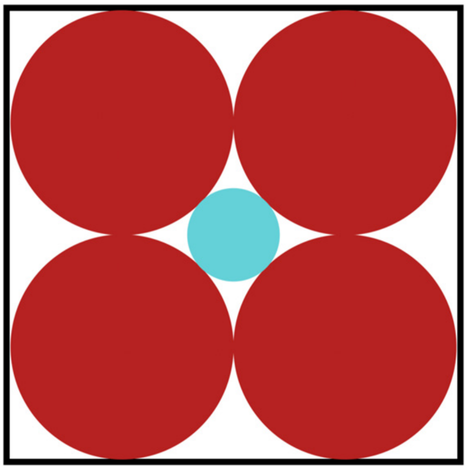


Figure 1: The 2D example.

In 3D we can pack a box with eight unit spheres (see Figure 2) and the smallest sphere which can sit in the very middle would have a radius of 0.73205. Which is slightly bigger than the 2D case. But our human intuition is that while there may be a bit more wriggle room with the extra dimensions, ultimately the central sphere will remain bounded.

This is absolutely not the case. In 4D the sixteen unit spheres in a box allow enough space for a central

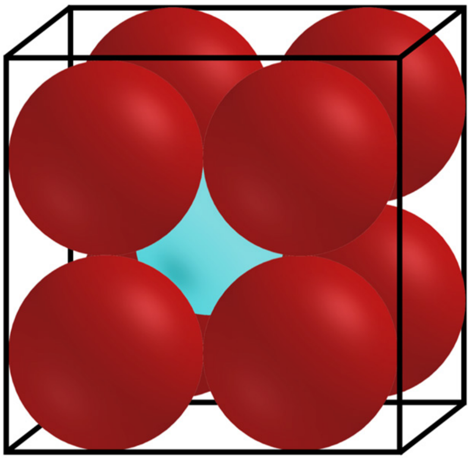


Figure 2: The 3D example.

sphere the same size as them: with a radius of 1. By 10D that central sphere is so big it reaches outside the box and by 26D it is twice as wide as the box.

The 'shape' and distribution of content within a higher dimension sphere does not match what we expect. In one (technically incorrect but still slightly illuminating) sense, spheres are more spiky than we think, with thin extremities which can reach through and pack around other objects. In another less incorrect sense: spheres in this high dimensions becomes all outer shell and very little internal, central volume.

Talagrand worked with shapes such as these, finding new and novel ways to put limits on where the bulk of a probability distribution could be.

The ubiquity of Gaussian processes means there are many possible applications of Talagrand's work. One specific one is the condensed matter physics problem of a "spin glass". This is an arrangement of matter which is not a 'glass' like in a window, but rather a random structure of magnetic moments.

Spin glasses sit between the highly organised magnetic properties of ferromagnet materials and the random arrangement of paramagnetic materials. If a ferromagnetic substance is heated and an external magnetic field used to align all the internal magnetic moment, this neat alignment will persist even after the substance is cooled. A paramagnetic substance will exponentially lose its internal magnetic field once the external field is removed. A spin glass however, will also lose its own magnetic arrangement but at a rate and in a way which currently defies our understanding.

The internal magnetic structures within a spin glass are arranged randomly, but with some order which causes their complex behaviours.

Physicists developed theories and limits on how this randomness would behave. Talagrand was able to come in and, despite ignoring the physics, use mathematics to provide a solid proof that the limits were indeed correct.

Michel Talagrand work in probability theory and related areas has produced several key insights and proofs using novel techniques. These would be beautiful in their own mathematical right, but the direct application to physical systems makes these results extra meaningful.