1. On the Atiyah—Singer index theorem

I shall explain a little about what the Atiyah—Singer index theorem is, why it is important, and what it is useful for. Here is a brief statement:

> **Theorem** (M.F. Atiyah and I.M. Singer): *Let $P(f) = 0$ be a system of differential equations. Then*
>
> $$\textit{analytical index}(\mathrm{P}) = \textit{topological index}(\mathrm{P}) .$$

The word "theorem" (from the Greek "theorein", to look at, cf. "theater") means that this is a mathematically proved assertion that is worthy of closer examination. The result was announced in 1963 and published in 1968.

2. Introduction

Modern applications of mathematics usually start out with a **mathematical model** for a part of reality, and such a model is almost always described by a **system of differential equations**. To make use of the model one seeks the solutions to this system of differential equations, but these can be almost impossible to find. The critical new insight of Atiyah and Singer was that it is much easier to answer a slightly different question, namely: "How many solutions are there?" The Atiyah—Singer index theorem gives a good answer to this question, and the answer is expressed in terms of the **shape** of the region where the model takes place.

It is a point here that it is **not** necessary to find the solutions of the system to get to know how many solutions there are. And conversely, knowing the number of solutions can in fact make it easier to find these solutions.

A simple analogue can be to look at triangles and quadrangles in the plane. It can be complicated to find the angles in the corners of some of these figures, but sometime before Euclid someone realized that the sum of the angles in all the corners is always 180 degrees for a triangle, and 360 degrees for a quadrangle. The answer to this question is thus easily given, and depends only in a simple way on the shape of the figure, namely whether it has three or four vertices.

The study of functions, derivation and integration is called mathematical **analysis** (from Greek "analyein", to break up). The study of the corresponding simple information about the shape of the region where the model takes place (was it a triangle or a quadrangle?) is called **topology** (from Greek "topos", place). The names of these mathematical subfields give rise to the terms **analytical index** and **topological index** that we find in the index theorem.

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4. Applications of mathematics

In the beginning, mathematics was used to count (arithmetic), e.g. for bookkeeping, planning and trade, or to describe shapes (geometry), e.g. for measuring a plot of land, for cutting out fabric for a dress, or for building a bridge. Modern applications of mathematics are often concerned with modeling, and thereby predicting the development over time, of a complex, composite system, such as how oil and gas flow in porous rocks under the North Sea, how queues of text messages in a cellular network can best be resolved, or what the weather will be like this week-end.

Since Newton and Leibniz these mathematical models have almost always been described by a **system of differential equations**. To use mathematics for the intended application, one seeks to find the **solutions** of this system, The Atiyah—Singer index theorem is a fundamental insight that says that we can find out **how many solutions** the system has essentially by just knowing some simple, flexible pieces of information about the shape of the region being modeled. Even if the index theorem is a purely mathematical result, which links together analysis and topology, it can thus be used as a tool in almost all applications of mathematics.

5. A picture of the mathematical world

The subject of mathematics can coarsely be divided into four areas: algebra, analysis, topology and logic. Mathematics is a diverse language that can describe, discuss and model many different objects and problems, and the four areas tend to focus on different aspects of these objects. Nonetheless, there are no clear boundaries between these areas, and mathematics does also not live in isolation from other subjects.

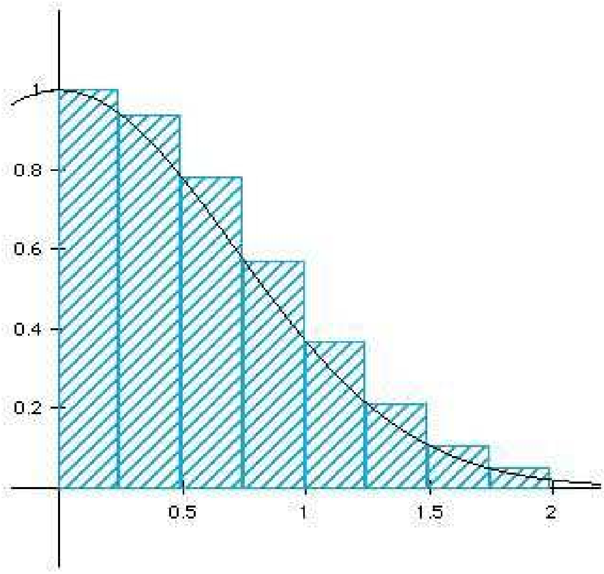
	economics	physics	sciences
computer science	**algebra**	**analysis**	applied mathematics

linguistics	**logic**	**topology**	medicine

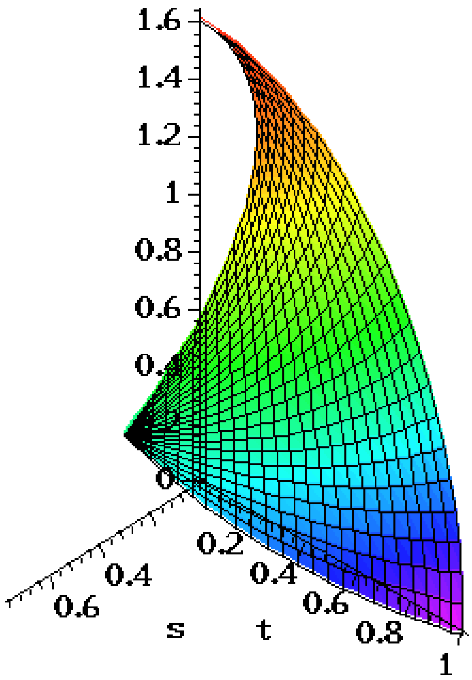
We shall here emphasize analysis and topology.

6. Analysis

In **analysis** an object is studied by first partitioning it into small pieces, and thereafter reassembling them (synthesis). Emphasis is put on the limiting case when the pieces become arbitrarily small, and simultaneously arbitrarily numerous. Keywords: differentiation, integration and calculus.



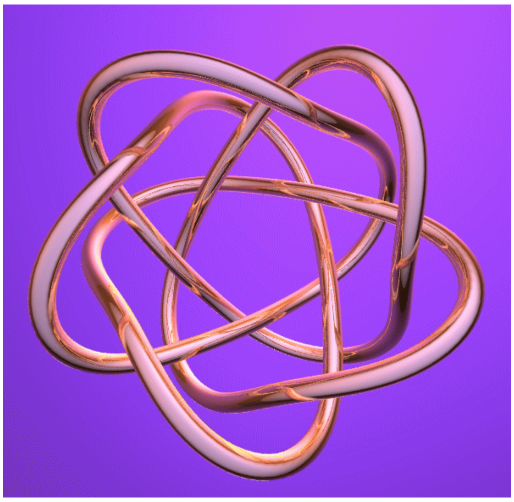
Area under a curve

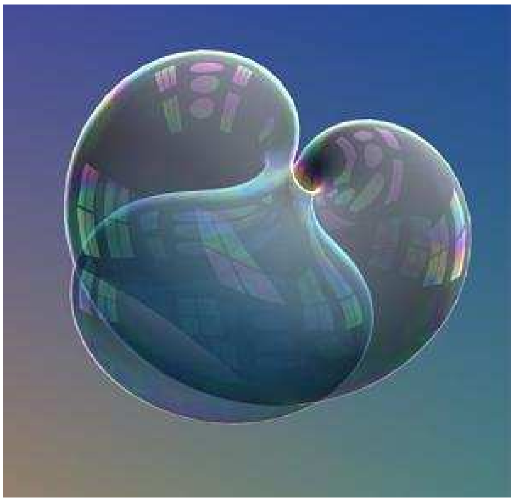


A sail?

7. Topology

In **topology** one studies how an object can have a shape, or a spatial aspect. In particular one emphasizes properties of the whole global shape, rather than the local appearance. If the shape is described by some notion of distance, then we usually talk about **geometry**.



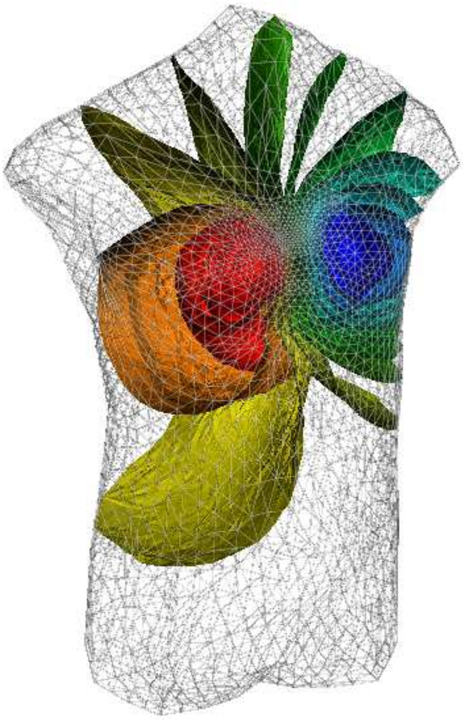


M.Thistlethwaite: "Symmetric knot"
G. Francis, J. Sullivan and S. Levy: "Spherical eversion"

8. Mathematical models

A mathematical model is an attempt to describe (part of) reality in mathematical language. One can also attempt to describe reality on ordinary language, but the mathematical language has the advantage that one can argue and reason with it in a completely precise and indisputable fashion. Therefore one can pursue a chain of thoughts in mathematical language through very many steps and still expect that the conclusion "about reality" is correct.

A mathematical model usually takes place in some spatial domain, or region, which we call X. This mathematical "space" can very well correspond both to space and time in the physical sense. For example, in a meteorological model a point in X can correspond to a particular place in the atmosphere above the northern hemisphere at a particular time during the coming week. In a medical model for the electrical impulses that regulate the heart, a point in X can correspond to a small part of the body at a specific time in the course of a series of heartbeats.



A torso An electrocardiogram (ECG)

The state of the model is described by a series of numbers for each point of X, e.g. the temperature, air pressure, humidity, wind speed etc. at the particular place in the atmosphere and the particular time. Mathematically this state is described by a series of **functions** f defined on the space X. In the medical model such a function can indicate the electrical field strength in the various regions of the body at the various times. The colored surfaces in the image on the left show regions with the same electrical field strength, near the beginning of a heartbeat.

9. Systems of differential equations

The physical laws that govern how the temperature, air pressure etc. (resp. the field strength) will change are well-known as long as one is only considering a small region in X, i.e., only looks as a small part of the atmosphere (resp. the body), for a short span of time. These laws can be expressed as a collection of equations, i.e., a system of equations.

These equations involve the functions f that describe the state of the model, but also the **derivatives** $f'(x) = df/dx$ of these functions. These are new functions which express how that state changes, either from one place to another, or from one time to another.

The derived functions are also called differentials, and therefore such a collection of equations is called a **system of differential equations**. Such a system can be briefly expressed in the form $P(f) = 0$. A series of functions f on X that are such that all the equations are satisfied describe a physically possible state, and are called a **solution** of the system of equations.

In nearly all the manifold applications there are of mathematical modeling, one wishes to know something about the solutions to such systems of differential equations.

((Show an animation that illustrates the spreading of electrical impulses from the heart. One wishes to unwind the development, in reverse, to be able to reconstruct what happens in the heart based on the measurements on the skin. This would be helpful for diagnosis.))

10. Analytical index

In practice it is usually very difficult to find precise formulas for the solutions. Therefore it is necessary to proceed in steps towards such an answer.

The first thing to know is whether there are some solutions at all. If not, something must probably be changed in the model. Thereafter one would like to know if there is one or many solutions, and if there are many, one would like to know something about how many there are.

The following definition is central.

Let $P(f) = 0$ be a system of differential equations for a model that is described by functions f on a space X. The **analytical index** of the system is an integer which essentially is the **number of solutions** to this equation. More precisely,

$$\text{analytical index}(P) := \dim \ker(P) - \dim \operatorname{coker}(P)$$

is equal to the number of parameters needed to describe all the solutions of the equation, minus the number of relations there are between the expressions $P(f)$.

For a concrete application this number is the first one needs to know about the solutions to the system of differential equations. If, for example, the analytical index is positive, then we **know** that the system has interesting solutions.

At first glance it is just as difficult to find the analytical index of a system as to find all the solutions. But the Atiyah—Singer index theorem tells us that, no, in fact it is easier to find the number of solutions to the system (i.e., to compute the analytical index) than it is to find the solutions themselves. All one needs to know is the shape, or topology, of the region where the model takes place.

11. The Atiyah—Singer index theorem, revisited

Here is a more precise statement of the index theorem.

Theorem (M.F. Atiyah and I.M. Singer):
Let $P(f) = 0$ be an elliptic system of partial differential equations defined over a closed, smooth, oriented n-dimensional manifold X. Then

$$\text{analytical index}(P) = \text{topological index}(P)$$

is given by the following explicit formula:

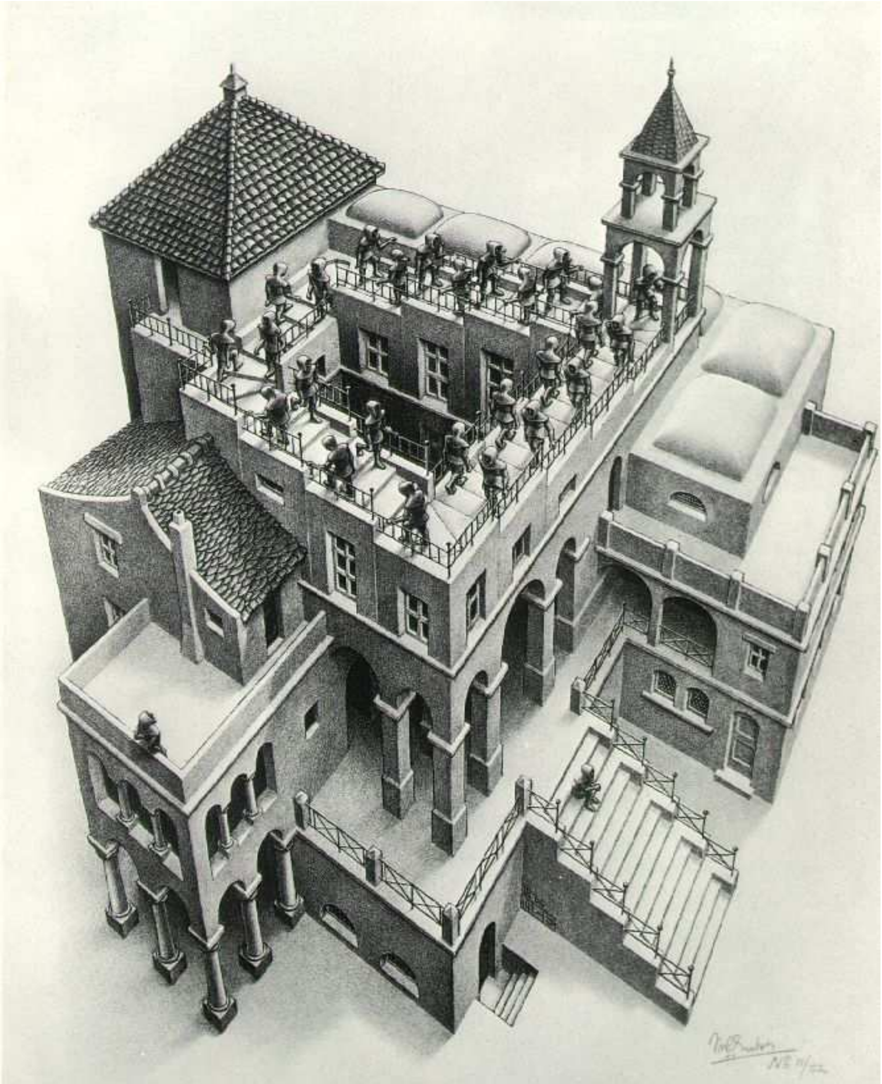
$$\text{topological index}(P) := (-1)^n < ch\,(s(P))\,.\,td\,(T_{\mathbb{C}}X)\,,\,[X] >$$

Here

- n is the dimension of the space X
- s(P) is the symbol of the system P
- ch is the Chern character
- $T_{\mathbb{C}}X$ is the complexified tangent bundle of X
- td is the Todd class
- . is the cup product
- [X] is the fundamental class of X, and
- <-,-> is the Kronecker pairing.

The ingredients in this formula are conceptually complex, but not harder to compute than that they can be manipulated directly by a mathematician. The expression essentially only depends on the shape, i.e., the topology, of the space X over which the equations take place.

12. A non-elementary example



M.C. Escher, "Treppauf und treppab"

A wanderer goes around, up or down a staircase. Here the spatial form X is a square, while the state is the function f defined so that $f(x)$ equals the height above the ground at each point x on X. The differential equation $f'(x) = 0$, where $P(f) = df/dx$, has a 1-dimensional space of solutions, namely the constant functions $f(x) = C$. The topological index of P over the square X equals 0, so by the index theorem, also the analytical index of P equals 0. There is therefore precisely $1 - 0 = 1$ relation between the possible expressions $f'(x)$, namely that the integral of $f'(x)$ over X equals 0. In particular the wanderer cannot ascend all the time, because then $f'(x) > 0$ and the integral of $f'(x)$ over X would be strictly positive.

13. Historical remarks

The Atiyah—Singer index theorem has historical predecessors, such as the Riemann—Roch formula in algebraic geometry and the signature theorem of F. Hirzebruch, and unifies these completely. It therefore has great intellectual and aesthetic value.

I.M. Gelfand conjectured ca. 1960 that the analytical index could have a purely topological description, but it was Atiyah and Singer that discovered and proved the correct form of this description. In this work they made use of topological K-theory, which was a new topological tool developed by Atiyah and Hirzebruch, inspired by a corresponding algebro-geometric tool defined by A. Grothendieck. The proof of the index theorem also involves a joint contribution on topological K-theory by Atiyah and G. Segal.

The Atiyah—Singer index theorem was a key to a very fertile flowering of the exchange of ideas between mathematics and theoretical physics in the 1980's and 90's:

Mathematical methods derived from the index theorem were used in physics (E. Witten) to develop "string theory," which is an attempt to find a common explanation for (1) gravitation, which we understand at large scales by the theory of relativity, and (2) the other forces, such as electro-magnetism, which we understand at small scales by the theory of quantum mechanics.

Conversely, ideas from physics, such as the study of magnetic monopoles and short-lived "instanton" particles (S.K. Donaldson), were used in mathematics to discover new, exceptional properties of four-dimensional differentiable spaces.

14. Conclusion

The Atiyah—Singer index theorem is a purely mathematical result. It tells us that a fundamental question in analysis, namely how many solutions there are to a system of differential equations, has a concrete answer in topology. This insight provides a short-cut to getting to know whether such solutions exist or not. The theorem is valuable, because it connects analysis and topology in a beautiful and insightful way. It is practical, because it explains how the manifold applications there are of mathematical analysis can make good use of the spatial, or topological, structure that underlies the problem at hand.

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